Exponential Generating Functions

Def: The exponential generating function (egf) for the sequence $a_0, a_1, a_2, \dots$ is $\displaystyle\sum_{n\geq 0} \frac{a_n x^n}{n!}$.

Ex: $a_n = 1$ for all $n \longleftrightarrow$ egf e^x

$a_n = n! \longleftrightarrow$ egf $\frac{1}{1-x}$

This second example demonstrates the usefulness of egfs... the sequence $\{n!\}$ has an ordinary generating function (ogf), but this ogf is not closed form (it doesn't converge anywhere).

In general: egfs are good for quickly growing sequences.

Example 8.17 Suppose

$a_{n+1} = (n+1)(a_n - n + 1)$ if $n \geq 0$,

and $a_0 = 1$. Find the egf for $\{a_n\}$.

Solution: Let $A(x) = \displaystyle\sum_{n\geq 0} \frac{a_n x^n}{n!}$.

$$\underbrace{\sum_{n=0}^{\infty} a_{n+1} \frac{x^{n+1}}{(n+1)!}}_{A(x)-1} = \underbrace{\sum_{n\geq 0} a_n \frac{x^{n+1}}{n!}}_{x A(x)} - \underbrace{\sum (n-1)\frac{x^{n+1}}{n!}}_{x^2 e^x - x e^x}$$

So: $A(x) - 1 = x A(x) - x^2 e^x + x e^x$

$$A(x) = \frac{1}{1-x} + x e^x$$

$$= \sum_{n\geq 0} x^n + \sum_{n\geq 0} \frac{x^{n+1}}{n!}$$

$$= \sum_{n\geq 0} n! \frac{x^n}{n!} + \sum_{n\geq 1} \frac{n x^n}{n!}$$

So $a_n = n! + n$.

Example 8.19 $a_{n+1} = 2(n+1) a_n + (n+1)!$

$a_0 = 0$.

$$\sum_{n\geq 0} a_{n+1} \frac{x^{n+1}}{(n+1)!} = 2x \sum a_n \frac{x^n}{n!} + \sum x^{n+1}$$

$$A(x) = 2x A(x) + \frac{x}{1-x}$$

$$= \frac{x}{(1-x)(1-2x)}$$

$$A(x) = \sum_{n\geq 0} (2^n - 1) x^n.$$

$$a_n = (2^n - 1) n!$$

Example: Bell numbers, $B(n)$.

Recall that

$B(n) = \#$ set partitions of $[n]$.

So,

$$B(n) = \sum_{k=0}^{n} S(n,k)$$

But also, by considering the block that n lies in, we have

$$B(n) = \sum_{i=1}^{n} \binom{n-1}{i-1} B_{n-i}.$$

Define

$$F(x) = \sum_{n\geq 0} \frac{B(n) x^n}{n!}.$$

Note that

$$F'(x) = \sum_{n\geq 1} \frac{B(n) x^{n-1}}{(n-1)!},$$

simply the egf for the sequence with its first term removed.

We solve for this derivative:

$$
\begin{aligned}
F'(x) &= \sum_{n\geq 1} \frac{B(n)\, x^{n-1}}{(n-1)!} \\
&= \sum_{n\geq 1} \left[\sum_{i=1}^{n} \binom{n-1}{i-1} B(n-i) \right] \frac{x^{n-1}}{(n-1)!} \\
&= \sum_{n\geq 1} \sum_{i=1}^{n} \underbrace{\frac{x^{i-1}}{(i-1)!}}_{\text{looks like } e^x} \underbrace{\frac{B(n-i)\, x^{n-i}}{(n-i)!}}_{\text{looks like } F(x)} \\
&= e^x F(x).
\end{aligned}
$$

Therefore,

$$\frac{F'(x)}{F(x)} = e^x,$$

or in other words,

$$\left(\ln F(x)\right)' = e^x.$$

This shows that

$$\ln F(x) = e^x + C.$$

Since

$$F(0) = B(0) = 1,$$

we must have

$$C = -1,$$

so

$$F(x) = \sum_{n\geq 0} \frac{B(n)x^n}{n!} = e^{e^x - 1}.$$