Bootstrap Percolation

Begin with a matrix M of 0s and 1s.

At each stage, if two or more neihbors of an entry are 1s, that entry becomes a 1.

This process continues indefinitely.

(Do example)

(Look at MathWorld)

Typical percolation question: if the initial matrix is random, with probability of a $1 = p$, what happens?

(Note: Prof. Winkler teaches Math 100 in winter term — percolation.)

Shapiro & Stevens (1991):

What if the intial matrix is a permutation Matrix?

When does M_π fill up?

Ex: $\pi = 24378165$ does not fill up.

Ex: $\pi = 16724358$ does fill up.

How many fill up?

Towards a Characterization...

An <u>interval</u> in the permutation π is a set I of contiguous indices such that $\pi(I) = \{\pi(i) : i \in I\}$ is also contiguous.

Ex: [2 [4 3]] [7 8] [1] [6 5]

Ex: [1 [6 7] [[2 4 3] 5] [8]]

Every permutation $\pi \in S_n$ has n intervals of length 1 and 1 interval of length n. If π has no other intervals, then it is called <u>simple</u>.

Inflations

Given $\sigma \in S_m$ and nonempty permutations $\alpha_1, \dots, \alpha_m$, the <u>inflation</u>

$$\sigma[\alpha_1, \dots, \alpha_m]$$

is the permutation obtained by each entry $\sigma(i)$ by an interval in the same relative order as α_i.

Ex: $24378165 = 2413[132, 12, 1, 21]$

Ex: $16724358 = 12[1672435, 1]$
$= 12[1, 5613247]$.

Uniqueness

Every permutation except 1 is the inflation of a unique simple permutation of length at least 2.

__Proof__: Consider the maximal proper (i.e., $\neq$ whole thing) intervals of a permutation.

If two of these intersect, then their union must be the whole permutation. In this case the permutation is the inflation of 12 or 21.

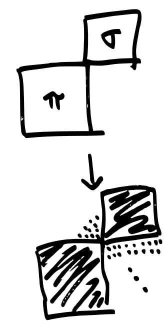
Otherwise, the maximal proper intervals are disjoint and, by maximality, define a simple permutation. ■

A sufficient condition

If π and σ both fill up, then $12[\pi, \sigma]$ and $21[\pi, \sigma]$ both fill up.

__Proof:__

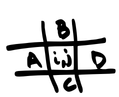
$12[\pi, \sigma] =$



Permutations that don't fill up

__Observation__: If σ of length ≥ 4 is simple, then M_σ does not fill up. In fact, bootstrap percolation leaves M_σ unchanged.

__Proof__: Suppose that the entry in position (i,j) is changed from 0 to 1 in the first iteration of bootstrap percolation. Then:

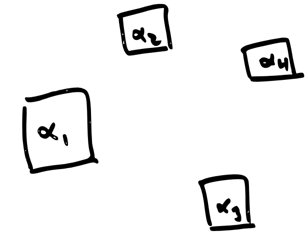


at least 2 of A, B, C, or D are filled in M_σ. If A is filled, then B or C must be filled. But this implies that σ isn't simple. ■

Extending this...

__Observation__: If σ does not fill up, then for any choice of nonempty $\alpha_1, \ldots, \alpha_m$, $\sigma[\alpha_1, \ldots, \alpha_m]$ also does not fill up.

__Proof__: Consider



. ■

Characterization

Theorem: The permutation π fills up under bootstrap percolation if and only if π can be built from the permutation 1 using the operations

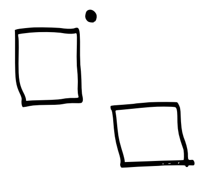
$$\sigma \oplus \tau = 12[\sigma, \tau] \quad \text{and}$$
$$\sigma \ominus \tau = 21[\sigma, \tau].$$

Def: These permutations are called separable.

How many are there?

Review from last time

132-avoiding permutations:

empty perm $\cup$ 

$\downarrow$

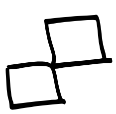
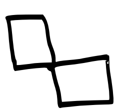
$$1 \quad + \quad f \times f$$

$\downarrow$

$$f = \frac{1 - \sqrt{1-4x}}{2x}.$$

Counting

Separable permutations:

empty $\cup$  $\cup$ 

$\downarrow$

?

Problem: the  and  decompositions aren't unique.

Solution: π is $\oplus$-indecomposable if it can't be written as $\sigma \oplus \tau$ for nonempty σ and τ.

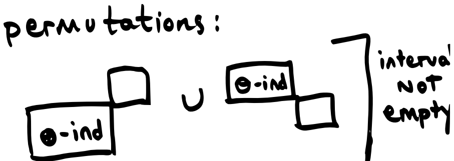
Analogous: $\ominus$-indecomposable.

Uniqueness

If π is $\oplus$-decomposable, then there is a unique $\oplus$-indecomposable permutation σ such that

$$\pi = \sigma \oplus \tau.$$

Separable permutations:

$\{1\}$ $\cup$ 

$\downarrow$

$$+ \quad g f \quad + \quad h f$$

where

f = g.f. for separables

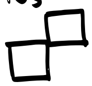
g = g.f. for $\oplus$-ind. separables

h = g.f. for $\ominus$-ind. separables

Note: none of these count empty perm.

Now note:

$g = \oplus$-ind. separables

$=$ separables $-$ 

$= f - gf$

So:

$$g(1+f) = f,$$

$$g = \frac{f}{1+f}.$$

Exactly the same:

$$h = \frac{f}{1+f}.$$

Therefore:

$$f = x + \frac{2f^2}{1+f}.$$

<u>Solving</u>

$$f^2 + f = x + xf + 2f^2$$

$$0 = f^2 + (x-1)f + x$$

$$f = \frac{1 - x \pm \sqrt{1 - 6x + x^2}}{2}$$

Which to choose?

Remember: we excluded the empty permutation, so $f(0) = 0$.

These are the <u>large Schröder</u> numbers.