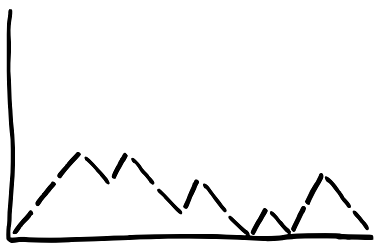
Dyck paths

A path in the plane from $(0,0)$ to $(2n, 0)$ which uses only the steps $(1,1)$ and $(1,-1)$ and never goes below the line $y=0$ is a Dyck path.



An example Dyck path

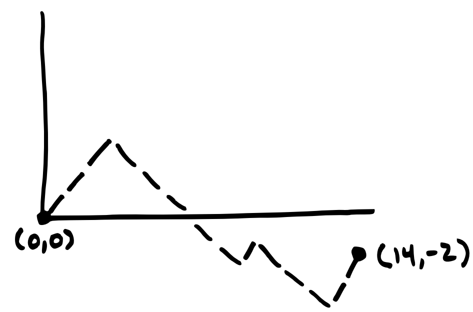
André's reflection principle (1881)

Total number of paths (not necessarily Dyck) from $(0,0)$ to $(2n, 0)$ with these steps:

$$\binom{2n}{n}. \rightarrow \text{choose up steps}$$

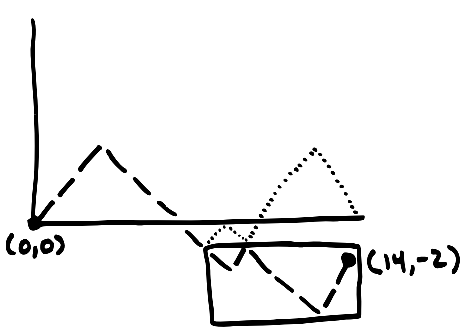
Number of paths from $(0,0)$ to $(2n, -2)$:

$$\binom{2n}{n-1}. \rightarrow \text{choose up steps}$$



Every path from $(0,0)$ to $(2n,-2)$ crosses the line $y=-1$ somewhere.

At the first such step, reflect the rest of the path about this line.



This yields a bijection:

$$\#\text{non-Dyck paths } (0,0)\to(2n,0) = \#\text{paths } (0,0)\to(2n,-2)$$

Therefore,

$$\#\text{Dyck paths} = \#\text{paths} - \#\text{non-Dyck} = \binom{2n}{n} - \binom{2n}{n-1}$$

Since
$$\binom{2n}{n-1} = \frac{(2n)!}{(n-1)!(n+1)!} = \frac{n}{n+1}\frac{(2n)!}{n!\,n!} = \frac{n}{n+1}\binom{2n}{n},$$

we see that

$$\#\text{Dyck paths} = \frac{1}{n+1}\binom{2n}{n}.$$

These are known as the Catalan numbers.

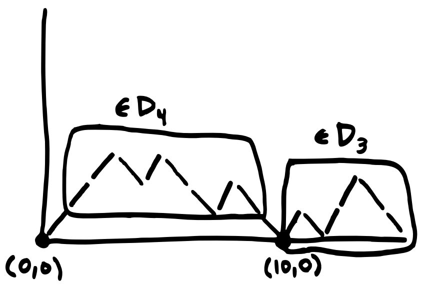
First passage decomposition

Now we know that

$$D_n = \#\text{Dyck paths } (0,0) \to (2n,0)$$
$$= \frac{1}{n+1}\binom{2n}{n}$$
$$= C_n \quad (\text{the } n^{th} \text{ Catalan } \#)$$

Is there an easier way?

Consider the first time the path hits the line $y=0$:



Example path from D_8

Therefore,

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}. \text{ for } n \geq 1$$

This should remind you of a product...

$$F(x) = \sum C_n x^n$$

Then the coeffient of x^n in F^2 is:

$$[x^n]\, F^2(x) = \sum_{i=0}^{n} C_i C_{n-i}.$$

So what we have above is the coeffient of x^ in $x F^2(x)$.

Therefore,

$$F(x) = x F^2(x) \underbrace{+1}_{\text{this accounts for } C_0}$$

To find F explicitly, we solve

$$x F^2 - F + 1 = 0$$

using the quadratic formula.

$$F = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

We need to know whether to take + or −. We decide by plugging in $x=0$. Since $F(0)=1$, we must take the −:

$$F = \frac{1-\sqrt{1-4x}}{2x}.$$

We could double-check this g.f. using Newton's Binomial Theorem; this is on the homework.

Algebraic generating functions

Last time we showed that a generating function is rational if and only if the sequence satisfies a linear recurrence relation.

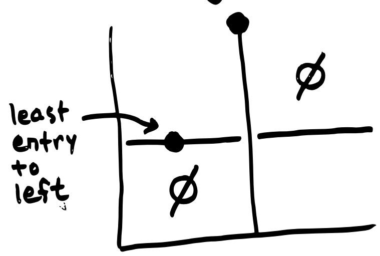
Today we would like to consider more general recurrences.

Recall that a complex number is algebraic if it is the root of a nonzero polynomial with rational coefficients.

The generating function $F(x)$ is algebraic if there is a nonzero polynomial $p(x,y)$ with rational coefficients such that $p(x, F(x)) = 0$.

The permutation π avoids 132 if there do not exist three indices $i<j<k$ such that $\pi(i) < \pi(k) < \pi(j)$.

Consider any 132-avoiding permutation. Draw its plot and look at its maximum entry:



This gives the Catalan recurrence!

The same number of permutations avoid 132 as avoid

231,
312, and
213.

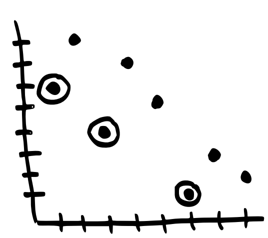
(why?)

Therefore the only other cases of length 3 to consider are 123 and 321, and these are the same (why?).

Lemma 14.8: The number of permutations of $[n]$ that avoid 123 is also C_n.

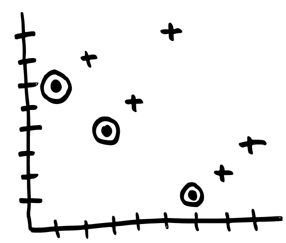
Proof: A left-to-right minima in π is an entry which is smaller than all those to its left.

Take a permutation π that avoids 123, and plot it, circling the left-to-right minima:

68475132 = 

Note that the remaining entries are in descending order.

Now remove everything but the left-to-right minima:



Thus we are missing the entries 8, 7, 5, 3, and 2. Insert these from left-to-right, at each stage inserting the smallest entry possible without creating a new left-to-right minima.

This is a bijection between 123- and 132-avoiding permutations.