Chapter 8: Generating functions

The term "generating function" is a misnomer.

① generating functions don't generate anything, and

② generating functions needn't be functions at all.

So what are generating functions?

Herb Wilf: "A generating function is a clothesline on which we hang up a sequence of numbers for display."

Def: The generating function for the sequence $a_0, a_1, a_2, \ldots$ is $\sum_{n \geq 0} a_n x^n$.

Math 8 examples:

$a_n = 1$ for all $n \longleftrightarrow$ g.f. $\frac{1}{1-x}$

$a_n = n \longleftrightarrow$ g.f. $\frac{x}{(1-x)^2}$

$a_n = n^2 \longleftrightarrow$ g.f. $\frac{x(1+x)}{(1-x)^3}$

$a_n = 3n^2 + 7 \longleftrightarrow ???$

Good conjectures to make now:

① G.f. for $a_n = n^k$ is $\frac{\text{something}}{(1-x)^k}$

② G.f. for polynomial sequence in $n \longleftrightarrow \frac{\text{poly}(x)}{(1-x)^k}$.

✱ On homework.

What good are generating functions?

Answer #1: They are how we solve recurrence relations.

Example from §8.1: The frog population of a lake grows by a factor of 4 each year, and at the end of every year, 100 frogs are removed. Assuming there were 50 frogs in the lake originally, how many frogs will be in the lake in n years?

Let a_n denote this quantity.

Recurrence: $a_{n+1} = 4a_n - 100$ for $n \geq 0$, $a_0 = 50$.

To get the g.f.:

$$\sum_{n \geq 0} a_{n+1} x^n = \sum_{n \geq 0} 4 a_n x^{n+1} - \sum_{n \geq 0} 100 x^{n+1}$$

Suppose $f(x) = \sum_{n \geq 0} a_n x^n$. Then:

$$f(x) - a_0 = 4x f(x) - \frac{100x}{1-x}.$$

Solving, we get

$$f(x) = \frac{a_0}{1-4x} - \frac{100x}{(1-x)(1-4x)}.$$

But what good is this?

The first sum is easy:
$$\frac{a_0}{1-4x} = \frac{50}{1-4x} = 50\sum_{n\geq 0}(4x)^n$$
$$= \sum_{n\geq 0} 50\cdot 4^n x^n.$$

For the second, we use __partial fractions__:

Find constants A and B so that
$$\frac{A}{1-x} + \frac{B}{1-4x} = \frac{100x}{(1-x)(1-4x)}$$
Multiplying by $(1-x)(1-4x)$ shows:
$$A(1-4x) + B(1-x) = 100x$$
$$(A+B) + (-4A-B)x = 100x,$$
so we need
$$\begin{cases} A+B = 0 \\ -4A-B = 100 \end{cases}$$

The solution is $A = \frac{100}{3}$, $B = \frac{-100}{3}$, so:
$$\frac{100x}{(1-x)(1-4x)} = \frac{100}{3}\frac{1}{1-4x} - \frac{100}{3}\frac{1}{1-x}$$
$$= \frac{100}{3}\left(\sum_{n\geq 0} 4^n x^n - \sum_{n\geq 0} x^n\right)$$
$$= \frac{100}{3}\left(\sum_{n\geq 0}(4^n-1)x^n\right).$$

Thus we have shown:
$$a_n = 50\cdot 4^n - 100\cdot\frac{4^n-1}{3}.$$

Steps:
① Define $f(x) = \sum_{n\geq 0} a_n x^n$.

② Transform recurrence into an equation for $f(x)$.

③ Solve for $f(x)$.

④ Find the coefficient of x^n in $f(x)$.

__Harder example__: Fibonacci numbers.

Recurrence: $f_n = f_{n-1} + f_{n-2}$ for $n\geq 2$,
$$f_0 = 1,$$
$$f_1 = 1.$$
Define $g(x) = \sum_{x\geq 0} f_n x^n$.

Thus
$$\sum_{n\geq 2} f_n x^n = \sum_{n\geq 2} f_{n-1} x^n + \sum_{n\geq 2} f_{n-2} x^n$$
$$g(x) - f_0 - f_1 x = x\,g(x) - f_0 + x^2 g(x), \text{ so}$$
$$g(x) = \frac{x}{1-x-x^2}.$$

The roots of $1-x-x^2$ are at
$$\varphi = \frac{1+\sqrt{5}}{2} \quad\text{and}$$
$$\alpha = \frac{1-\sqrt{5}}{2}.$$

Now using partial fractions, we can prove Binet's Formula.