Antichains in $(2^{[n]}, \subseteq)$

Sperner's Theorem (1928): If $\mathcal{A} \subseteq 2^{[n]}$ is an antichain, then $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Proof: We use the "LYM technique" of Lubell, Yamamoto, and Meshalkin from ~1954.

A chain is maximal if no other element can be added to it without destroying the chain property.
The poset $(2^{[n]}, \subseteq)$ clearly has $n!$ maximal chains

$$\emptyset = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n = [n].$$

Also, if A is a set of size k, then $k!(n-k)!$ of these maximal chains include A. Finally, if $\mathcal{A} \subseteq 2^{[n]}$ is an antichain, then every saturated chain can include at most one element of $\mathcal{A}$, so:

$$\begin{aligned} n! &= \#\text{ maximal chains} \\ &\geq \#\text{ maximal chains which include a member of } \mathcal{A}, \\ &= \sum_{A \in \mathcal{A}} |A|!\,(n-|A|)! \\ &= n! \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}}. \end{aligned}$$

This expression is minimized when $\binom{n}{|A|} = \binom{n}{\lfloor n/2 \rfloor}$, so we see that

$$|\mathcal{A}| \Big/ \binom{n}{\lfloor n/2 \rfloor} \leq 1. \blacksquare$$

An application of Sperner's Theorem

Littlewood-Offord (1943): Suppose $z_1, z_2, \ldots, z_n \in \mathbb{C}$ satisfy $|z_k| \geq 1$ for every k. Then the number of sums of the form $\sum_{k \in [n]} \varepsilon_k z_k$ where $\varepsilon_k \in \{\pm 1\}$ which lie inside the unit circle is at most

$$\frac{C\, 2^n \log n}{\sqrt{n}}$$

for some constant $C > 0$.

Erdős (1945): Suppose $x_1, x_2, \ldots, x_n \in \mathbb{R}$ satisfy $|x_k| \geq 1$ for every k. Then the number of sums of the form $\sum_{k \in [n]} \varepsilon_k x_k$ where $\varepsilon_k \in \{\pm 1\}$ which lie inside any open interval of length 2 is at most $\binom{n}{\lfloor n/2 \rfloor} \sim \frac{C\, 2^n}{\sqrt{n}}$.

Proof: Fix an open interval $I \subseteq \mathbb{R}$ of length 2. WLOG, we may assume that each x_k is positive. For any $A \subseteq [n]$, we define

$$S(A) = \sum_{k \in A} x_k - \sum_{k \notin A} x_k.$$

Note that the $S(A)$ quantities are precisely the sums we are interested in.

If $A \subsetneq B$ then $S(B) > S(A)$; in fact, $S(B) \geq S(A) + 2$. Therefore at most one can lie in I, so the collection

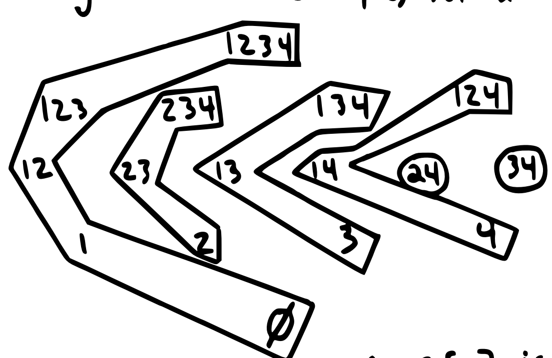
$$\mathcal{A} = \{A \subseteq [n] : S(A) \in I\}$$

forms an antichain. The result now follows from Sperner's Theorem. ∎

To get the full Littlewood-Offord result, we will need to generalize Sperner's Theorem. While the LYM technique gave a beautiful proof of Sperner's Theorem, we need to use a more combinatorial proof for the generalization. As an added bonus, this combinatorial proof also shows that the binomial coefficients are unimodal.

<u>Symmetric Chain Decompositions</u>

We begin with an example, for $2^{[4]}$:



A chain $A_1 \subset A_2 \subset \cdots \subset A_h \subseteq [n]$ is a <u>symmetric chain</u> if

① $|A_{k+1} \setminus A_k| = 1$ for all k, and

② $|A_1| + |A_h| = n$.

A <u>symmetric chain decomposition (SCD)</u> of $(2^{[n]}, \subseteq)$ is a collection of disjoint symmetric chains whose union is $(2^{[n]}, \subseteq)$.

It had been known that $(2^{[n]}, \subseteq)$ has an SCD since the early 1950's. The most explicit, and frequently used, SCD for $(2^{[n]}, \subseteq)$ is due to Greene and Kleitman from 1976.

To specify an SCD, we need to describe:

- when to stop
- how to go "up" if we don't stop

We do this by specifying a successor function

$$\sigma : 2^{[n]} \to 2^{[n]} \cup \{\text{stop}\}.$$

The set $A \subseteq [n]$ has <u>characteristic vector</u> $\chi(A)$ in which the k^{th} component is 1 if $k \in A$ and 0 otherwise.

Given $\chi(A)$, we <u>match</u> the 0's and 1's from left to right: when a 0 is encountered, it becomes an <u>unmatched 0</u>; when a 1 is encountered, it gets matched to the rightmost unmatched 0 (which becomes matched).

<u>Ex</u>: $A = \{3, 4, 6, 7, 8\} \subseteq [9]$

$\chi(A) = 0\,0\,1\,1\,0\,1\,1\,1\,0$ unmatched 0

unmatched 1s

Now we can describe the successor function:

$$\sigma(A) = \begin{cases} \text{stop} & \text{if no unmatched 0s} \\ A \cup \{k\} & \text{if } k \text{ is the leftmost unmatched 0} \end{cases}$$

<u>Ex</u>: $\sigma(\{3,4,6,7,8\}) = \{3,4,6,7,8,9\}$.

$\chi(\{3,4,6,7,8,9\}) = 0\,0\,1\,1\,0\,1\,1\,1\,1$

This has no unmatched 0s, so the chain stops here.

<u>Greene-Kleitman (1976)</u>: This construction gives an SCD for $(2^{[n]}, \subseteq)$.

<u>Proof</u>: Homework. ■

We now strengthen Sperner's Theorem:

<u>Katona and Kleitman (independent, 1965)</u>: Choose any partition X/Y of $[n]$ into nonempty parts, and a collection $\mathcal{A} \subseteq 2^{[n]}$. If there are no indices $j \neq k$ so that both

- $A_j \subset A_k$ and
- $A_k \setminus A_j \subseteq X$ or Y,

then $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

<u>Proof</u>: Consider SCDs for both X and Y. For any pair of chains

$C: E_1 \subset E_2 \subset \cdots \subset E_g$ and

$D: F_1 \subset F_2 \subset \cdots \subset F_h$,

we form the "symmetric rectangle"

$$\begin{array}{cccc} E_1 \cup F_1 & E_1 \cup F_2 & \cdots & E_1 \cup F_h \\ E_2 \cup F_1 & & & \\ \vdots & & & \vdots \\ E_g \cup F_1 & & \cdots & E_g \cup F_h. \end{array}$$

If $\mathcal{A}$ satisfies the hypotheses, then we can have at most one member of $\mathcal{A}$ in any row or column, so the number of members of $\mathcal{A}$ which occur above is bounded by $\min(g, h)$.

By homework problem #7, $\min(g,h)$ is precisely the number of subsets of size $\lfloor n/2 \rfloor$ which occur in this rectangle.

Therefore, by considering all such symmetric rectangles, we see that

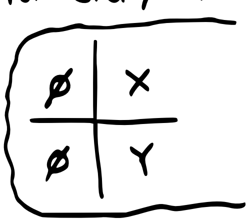
$$|\mathcal{a}| \leq \binom{n}{\lfloor n/2 \rfloor}. \blacksquare$$

Finally, we return to the Littlewood-Offord problem.

<u>Katona and Kleitman (independent, 1965)</u>:
Suppose $z_1, z_2, \ldots, z_n \in \mathbb{C}$ satisfy $|z_k| \geq 1$ for every k. Then the number of sums of the form $\sum_{k \in [n]} \varepsilon_k z_k$ where $\varepsilon_k \in \{\pm 1\}$ which lie inside the unit circle is at most $\binom{n}{\lfloor n/2 \rfloor}$.

<u>Proof</u>: WLOG, we may assume that $\mathrm{Re}\, z_k$ is nonnegative for every k. Set

$$X = \{k : \mathrm{Im}\, z_k \geq 0\}$$
$$Y = \{k : \mathrm{Im}\, z_k < 0\}$$



For $A \subseteq [n]$, we define

$$S(A) = \sum_{k \in A} z_k - \sum_{k \notin A} z_k.$$

If $A \subsetneq B$ and $B \setminus A \subseteq X$, then $S(A)$ and $S(B)$ differ by a set of complex numbers all in the first quadrant and all of norm at least 2, so at most one of $S(A)$ and $S(B)$ can lie inside the unit circle.

The same holds if $A \subsetneq B$ and $B \setminus A \subseteq Y$, so the collection

$$\mathcal{a} = \{A \subseteq [n] : |S(A)| < 1\}$$

satisfies the hypotheses of the previous theorem, showing that

$$|\mathcal{a}| \leq \binom{n}{\lfloor n/2 \rfloor}. \blacksquare$$