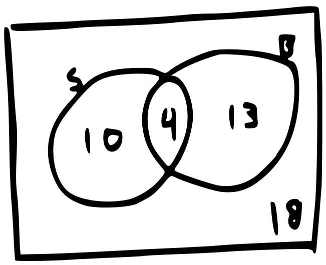
Mobiüs Inversion (and Inclusion-Exclusion)

Ex 7.1: There are...

45 students
14 play (S)occer
17 play (B)asketball
4 play S & B.

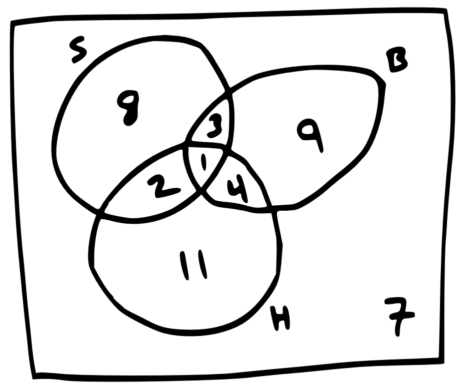
How many play neither?

Soln:



Ex 7.2

45 students
14 play (S)occer,
17 play (B)asketball,
18 play (H)ockey,
4 play S & B,
3 play S & H,
5 play B & H,
1 plays S, B, & H.



A bit of formalism.

Define $g: 2^{\{S,B,H\}} \to \mathbb{N}$

by $g(X)$ = # students who play all sports in X (but possibly more)

Also define $f: 2^{\{S,B,H\}} \to \mathbb{N}$

by $f(X)$ = # students who play precisely the sports in X.

Note: $g(X) = \sum_{Y \supseteq X} f(Y)$.

Inclusion-Exclusion: $f(\emptyset) = \sum_{Y \supseteq \emptyset} (-1)^{|Y|} g(Y)$.

So we are "inverting" a function which maps from $2^{\{S,B,H\}}$ to $\mathbb{N}$.

But why not generalize?

Consider functions from any poset to any ring.

In number theory, if

$$g(n) = \sum_{d|n} f(d)$$

then

$$f(n) = \sum_{d|n} \mu\left(\tfrac{n}{d}\right) g(d)$$

where

$$\mu\left(\tfrac{n}{d}\right) = \begin{cases} (-1)^t & \text{if } \tfrac{n}{d} \text{ is the product of } t \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

μ is called the Mobiüs function.

Instead of $\mu(\frac{n}{d})$, we'll consider $\mu(d, n)$.

Really, this is just linear algebra. Returning to example 7.2, let's consider matrices & vectors indexed by subsets of $\{S, B, H\}$. We need to pick a standard order. Let's choose

$\emptyset$, S, B, H, SB, SH, BH, SBH.

Then...

$$\vec{g} = \begin{bmatrix} 45 \\ 14 \\ 17 \\ 18 \\ 4 \\ 3 \\ 5 \\ 1 \end{bmatrix} \qquad \vec{f} = \begin{bmatrix} 7 \\ 8 \\ 9 \\ 11 \\ 3 \\ 2 \\ 4 \\ 1 \end{bmatrix}$$

What's the relationship?

$$\vec{g} = \begin{array}{cc} & \begin{matrix} \emptyset & S & B & H & SB & SH & BH & SBH \end{matrix} \\ \begin{matrix} \emptyset \\ S \\ B \\ H \\ SB \\ SH \\ BH \\ SBH \end{matrix} & \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\zeta} \end{array} \vec{f}$$

More generally, for any poset P, the <u>Incidence algebra $I(P)$</u> is the set of all matrices M indexed by elements of P such that $M(x,y) = 0$ unless $x \le y$.

We want to invert

$$\zeta(x,y) = \begin{cases} 1 & \text{if } x \le y \\ 0 & \text{otherwise,} \end{cases}$$

The invese, $\mu(x,y)$, is called the Mobiüs function of P.

While we colld use row operation (or something else) to invert, there is a better way.

$\mu \zeta = \mathrm{Id}$

Look at its (x,y) entry:

$$\sum_{z} \mu(x,z)\,\zeta(z,y)$$

$$= \sum_{z \le y} \mu(x,z)$$

Would like μ in $I(P)$, so...

$$= \sum_{x \le z \le y} \mu(x,z)$$

$$= \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

This shows that we can define μ inductively.

$$\mu(x,x) = 1$$
$$\mu(x,y) = 0 \text{ if } x \not\le y$$

Otherwise, $x < y$, so

$$0 = \sum_{x \le z \le y} \mu(x,z)$$
$$= \sum_{x \le z < y} \mu(x,z) + \mu(x,y),$$

so we can recursively define

$$\mu(x,y) = -\sum_{x \le z < y} \mu(x,z).$$

The Principle of Mobiüs Inversion

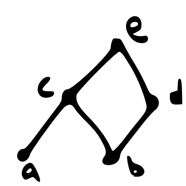
Suppose f and g are functions from the poset P to any ring and satisfy

$$g(x) = \sum_{y \ge x} f(y)$$

for all $x \in P$. Then

$$f(x) = \sum_{y \ge x} \mu(x,y)\, g(y).$$

Proof: $\vec{g} = \zeta \vec{f}$, so $\vec{f} = \mu \vec{g}$. $\square$

Ex: $P =$ 

Fix an ordering on the vertices of P. Preferrably, this should be a linear extension.

a, b, c, d, e.

Then

$$\mu = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 0 & & 0 & \\ 0 & 1 & & & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Now fill in the rest.