EXERCISES FOR WEEK 1

1. In these three numerical problems find as simple a solution as possible.

(a) How many subsets of the set $[10] = \{1, 2, \ldots, 10\}$ contain at least one odd integer?

(b) In how many ways can seven people be seated in a circle if two arrangements are considered the same whenever each person has the same neighbors (not necessarily on the same side)?

(c) How many compositions of 19 use only the parts 2 and 3?

2. Give at least two substantially different proofs that for all positive integers n,

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$

3. Prove that for $1 \leqslant k < n$, the part k occurs a total of $(n-k+3)2^{n-k-2}$ times among all the 2^{n-1} compositions of n. For example, if $n = 4$ and $k = 2$, then the part 2 occurs once in $2+1+1$, $1+2+1$, and $1+1+2$, and twice in $2+2$, for a total of $5 = (4-2+3)2^{4-2-2}$ times.

4. Compute the polynomial $p^{3,3}$, counting compositions that fit in a 3×3 rectangle.

5. Give an example of a log-concave polynomial which does not have real roots. (Therefore, the converse of Newton's Real Roots Theorem is false.)

6. Give an example of two unimodal polynomials whose product is not unimodal.

The *Fibonacci numbers* are defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 3$. Exercises 7–9 concern the Fibonacci numbers. Provide justification for your answers.

7. Let a_n denote the number of subsets S of the set $[n] = \{1, 2, \ldots, n\}$ such that S contains no two consecutive integers. Express a_n in terms of the Fibonacci numbers.

8. Express the number of compositions of n into parts equal to 1 or 2 in terms of the Fibonacci numbers.

9. Express the number of compositions of n into parts greater than 1 in terms of the Fibonacci numbers.