EXERCISES FOR FINAL EXAM

These exercises constitute the take-home final exam in Math 68.

- Solutions are due by the end of the day on Friday, December 11th.
- *But*, please note that I will be out of town from Wednesday, December 9th until Friday, December 11th and therefore unable to offer any assistance.
- You may consult reference books and the interwebs, but please do *not* copy answers from these sources.
- This exam is untimed.
- Your solutions to this exam will be graded on a finer scale than regular homework problems, more like your solutions to the midterm exam were graded.
- Therefore, please make sure that the steps in your solutions are *explained*.

1. Find an explicit formula for a_n if

$$a_{n+2} = 5a_{n+1} - 6a_n + 2n - 1$$

for $n \geqslant 0$ with initial conditions $a_0 = 0$ and $a_1 = 1$. *Note:* here and in other problems, you are allowed to use a computer algebra program, providing that you also turn in its output.

2. Count unlabeled graphs on 5 vertices by their number of edges using Póyla's Theorem (like we did in Lecture 25). Include a chart describing the action of S_5 on edges.

3. Calculate the number of spanning trees of the 6×6 torus graph G. The vertices of this graph are ordered pairs in $\mathbb{Z}_6 \times \mathbb{Z}_6$, where (i, j) is connected to $(i, j-1)$, $(i, j+1)$, $(i-1, j)$, and $(i+1, j)$, mod 6.

4. Find an explicit formula for the number of trees on n labeled nodes with exactly 4 leaves. *Hint:* The answer involves Stirling numbers, although there may be other ways to do the problem. *Second hint:* You may want to investigate Prüfer codes.

5. Consider expanding the product

$$\prod_{1 \leqslant i < j \leqslant n} (x_i + x_j)$$

into monomials. Prove that the number of monomials in this expansion equals the number of forests on n labelled vertices. For example there are 2 forests on 2 labeled vertices, which correspond to the monomials x_1 and x_2, while there are 7 forests on 3 labeled vertices, corresponding to the 7 terms of the expansion of $(x_1 + x_2)(x_1 + x_3)(x_2 + x_3)$:

$$x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + 2x_1 x_2 x_3.$$