1. Chapter I: #10ab.

ANS: This is a test to see if you can parse a definition precisely. Recall that a function $f : A \to B$ is a subset $f \subset A \times B$ such that for *every* $a \in A$, there is a unique $b \in B$ such that $(a, b) \in f$. If *either* A or B is empty, then so is $A \times B$. Hence there is only one subset of $A \times B$ — namely the empty set $\emptyset$. The only question is whether or not the empty set is a function.

(a) If A is nonempty and $B = \emptyset$, then given $a \in A$, there can be no $(a, b) \in A \times B = \emptyset$, so there are no functions $f : A \to \emptyset$.

(b) On the other hand, if $A = \emptyset$, then whether or not B is empty, the condition for all $a \in A$ there is a unique $b \in B$ such that $(a, b) \in \emptyset$ is vacuously satisfied. Hence the empty set is a function, and the only function, from $\emptyset$ to B.

2. Chapter II: #11.

ANS: Here we have to show that if $a > 1$, then $\{a, a^2, a^3, \dots\}$ is not bounded. Ok, suppose not. Then there is $x \in \mathbf{R}$ such that $a^n \le x$ for all $n \in \mathbf{N}$.

Next we turn to the hint. We'll show that

$$\Big(1 + \frac{1}{n}\Big)^n \ge 2 \quad \text{for all } n \in \mathbf{N} \tag{†}$$

using induction.[1] Let A be the subset of $\mathbf{N}$ for which (†) holds. Clearly $1 \in A$ and if $n \in A$, then

$$\begin{aligned}\Big(1 + \frac{1}{n}\Big)^{n+1} &= \Big(1 + \frac{1}{n}\Big)^n \Big(1 + \frac{1}{n}\Big) \\ &\ge 2 \cdot 1 = 2.\end{aligned}$$

(Here we've used $n \in A$ and $(1 + \frac{1}{n}) \ge 1$.) This shows (†) holds for all n.

Next I claim that

$$2^k \ge k \quad \text{for all } k \in \mathbf{N}. \tag{‡}$$

Again, we'll use induction. Let A be the set of k for which (‡) holds. Clearly $1 \in A$. Suppose $n \in A$. Then

$$2^{n+1} \ge 2^n(2) \ge 2n = n + n \ge n + 1.$$

Thus (‡) holds for all k. But there is a $k > x$ (by LUB 1). Since $a - 1 > 0$, there is a $n \in \mathbf{N}$ such that $\frac{1}{n} < a - 1$ and

$$a < \big(1 + \frac{1}{n}\big).$$

[1]Alternately, you could use the result we proved in lecture that $(1 + x)^n \le 1 + nx$ provided $x \ge -1$. You can't for example, use the binomial theorem as we haven't proved it. Of course, you could prove it and then use it.

But then

$$\begin{aligned} a^{kn} &> \left(\left(1+\frac{1}{n}\right)^n\right)^k \\ &\ge 2^k \\ &\ge k \\ &> x. \end{aligned}$$

But this contradicts our choice of x. This finishes the proof.

3. Chapter II: #13.

ANS: Since each S_i is nonempty and bounded above, each set has a least upper bound s_i. Define

$$S_1 + S_2 = \{\, x + y : x \in S_1 \text{ and } y \in S_2 \,\}.$$

We are supposed to show that $\mathrm{lub}(S_1 + S_2) = s_1 + s_2$. But if $x \in S_1$ and $y \in S_2$, then

$$x + y \le s_1 + s_2.$$

Hence $S_1 + S_2$ is bounded above (as well as nonempty). Hence $S_1 + S_2$ at least has an least upper bound. Since $s_1 + s_2$ is an upper bound, it will suffice to see that $s_1 + s_2 - \epsilon$ is not an upper bound for any $\epsilon > 0$. But $s_1 - \epsilon/2$ can't be an upper bound for S_1. Thus there is a $t_1 \in S_1$ such that $t_1 > s_1 - \epsilon/2$. Similarly, there is a $t_2 \in S_2$ such that $t_2 > s_2 - \epsilon/2$. But now we have $t_1 + t_2 \in S_1 + S_2$ and

$$t_1 + t_2 > s_1 + s_2 - \epsilon.$$

Thus $s_1 + s_2 - \epsilon$ is not an upper bound and we're done.