Final Exam, take-home part, Math 63, Fall 2003
due Tuesday, December 9, 2003, 8AM

Rules of the game:

(1) Solve **one** of the three topics included.
(2) You can work with your favorite group (no more than four people in the group). This group can be different from the one that you had regularly during the term. Each student should turn in an individual answer. Copying is prohibited. You should indicate if you worked in group and who were the other members of the group. Live support from outside the group can come from God and from your instructor only. You are allowed to use the resources from the library and the internet. If asked, you should be able to reproduce and explain the answers that you turn in.
(3) Neatness and clarity count. If you can go beyond what is requested (give significant examples, applications etc.) you will receive bonus points.
(4) Use your imagination. You are encouraged to go beyond the suggested steps of each project and try your own ideas when solving the problems. Try to impress me! Show off! This is a time when you have to look back and make a synthesis of what you have learnt during the term.
(5) Make a bibliography at the end of the paper. You must acknowledge all the sources you used, including books, `www` locations, your friends, your enemies, your instructor etc. If in doubt on what to do next with the exam, your instructor must be the main source of advice.
(6) Use only one side of each sheet of paper.
(7) The take-home exam is due Tuesday, December 9, 8AM, before the written final exam. It represents half of the final exam (10% of the course grade).

The exams will be graded both on the mathematical correctness and on the clarity of exposition. Starting with the next page you will find the list of topics.

1 The Characterization of Riemann Integrability

The goal is to prove the following part of Lebesgue's theorem:

> *If the bounded function $f : [a, b] \to \mathbb{R}$ is continuous almost everywhere then it is Riemann integrable.*

(**1**) Recall first how you proved that a monotone function is Riemann integrable. Then recall the arguments involved in solving the Exercises 14.9, 14.10, and 14.11 of Handout #26. How where the discontinuities of a function handled?

(**2**) Recall the definition of the *oscillation of* f *at* x, $\operatorname{osc}(f, x)$. Show that f is continuous at x if and only if $\operatorname{osc}(f, x) = 0$.

(**3**) What can you say (open/closed, size etc.) about the set

$$A_\varepsilon = \{\, x \in [a, b] \mid \operatorname{osc}(f, x) \geq \varepsilon \,\}?$$

(**4**) Show that you can find a partition P such that $U(f, P) - L(f, P) < \varepsilon$. (You have to cover A_ε in a certain way, and the complement of this cover in another 'controlled' way.) This will prove the theorem.

(**5**) Consider next a bounded function $f : [a, \infty) \to \mathbb{R}$. We say that f is Riemann integrable if the limit

$$\lim_{b \to \infty} \int_a^b f(x)\, dx$$

exists. (In particular, it is assumed that f is Riemann integrable over each finite interval $[a, b]$.) The limit is denoted by $\int_a^\infty f(x)\, dx$. Does your proof of the Lebesgue theorem generalize to such functions? If yes, prove it; if no, give an example of a function for which the theorem of Lebesgue fails.

2 Existence of Nowhere Differentiable Functions

The goal is to prove that

> *there are continuous functions that are not differentiable at any point in the domain of definition.*

(**1**) For simplicity we shall assume that all the functions that we consider are defined on $[0, 1]$. Prove that the normed vector space $C([0, 1])$, of all continuous functions $f : [0, 1] \to \mathbb{R}$ with supremum norm, is complete.

(**2**) Let

$$f_1 : [0,1] \to \mathbb{R}, \quad f_1(x) = \begin{cases} x - \frac{1}{2} & , \quad \text{if } x \in [\frac{1}{2}, \frac{3}{4}], \\ -x + 1 & , \quad \text{if } x \in [\frac{3}{4}, 1], \\ 0 & , \quad \text{elsewhere.} \end{cases}$$

Extend f_1 by periodicity on the entire $\mathbb{R}$, and still call the extension f_1. For $n \geq 2$ let:

$$s_n : [0,1] \to \mathbb{R}, \quad s_n(x) = f_1(x) + \frac{1}{2} f_1(2x) + \frac{1}{2^2} f_1(2^2 x) + \dots \frac{1}{2^{n-1}} f_1(2^{n-1} x).$$

Prove that $s = \lim_{n\to\infty} s_n$ is a continuous function.

(**3**) By an explicit computation (using the definition) show that s is not differentiable an any point in $[0, 1]$. (*Hints.* Using the binary expansion of real numbers may be helpful. You can show that, for every $x_0 \in [0, 1]$, there are sequences $(x_n)_n$ and $(y_n)_n$, both convergent to x_0, satisfying

$$\lim_{x_n \to x_0} \frac{s(x_n) - s(x_0)}{x_n - x_0} \neq \lim_{y_n \to x_0} \frac{s(y_n) - s(x_0)}{y_n - x_0}.$$

Or something like this ...)

3 The Fundamental Lemma of Analysis on $\mathbb{R}$

The goal is to show that there is a **unifying principle of the analysis on** $\mathbb{R}$: **one** method that makes easy to prove **most** of the important theorems on $\mathbb{R}$ — the Intermediate Value Theorem, the Mean Value Theorem, the Darboux property of continuous functions, the Riemann integrability of continuous functions etc.

(**1**) Let A be a nonvoid set. A *relation* ρ *on* A is a subset R_ρ of $A \times A$. R_ρ represent those pairs (a, b), $a, b \in A$, which 'are in relation ρ.' We denote this by $a \,\rho\, b$, and we say that 'a is in relation ρ with b.' Consider $A = \mathbb{R}$ and ρ to be the strict inequality $>$ between real numbers. What is R_ρ in this case?

(**2**) A relation ρ is called *transitive* if from $a \,\rho\, b$ and $b \,\rho\, c$ it follows that $a \,\rho\, c$. Give two examples of transitive relations.

(**3**) Prove the following:

> **Fundamental Lemma of Analysis on $\mathbb{R}$.** *Let ρ be a transitive relation on the interval $[a, b]$. If each $x \in [a, b]$ has a neighborhood N_x such that $u \,\rho\, v$ whenever $u \in [a, x] \cap N_x$ and $v \in [x, b] \cap N_x$, then $a \,\rho\, b$.*

(**4**) Prove Cantor's Nested Intervals Lemma 2.6.2 using the Fundamental Lemma of (3). (*Hint.* Consider on $I_1 = [a_1, b_1]$ the relation:

$$u \,\rho\, v \quad \Leftrightarrow \quad u \leq v \text{ and } (\exists n \in \mathbb{N}) \text{ s.t. } [u, v] \cap [a_n, b_n] = \emptyset.)$$

(**5**) As you noticed in (4) the delicate part is finding the right transitive relation ρ. Prove the Intermediate Value Theorem using the Fundamental Lemma of (3).