7. $\mathbb{R}^n$ — NORMED VECTOR SPACE AND METRIC SPACE

Exercise 7.1. Cauchy-Schwarz inequality.

(a) Recall first the conditions that need to be imposed on the coefficients a, b, and c of the polynomial $f(t) = at^2 + bt + c$ in order to have $f(t) \geq 0$, for all $t \in \mathbb{R}$.

(b) Considering

$$f(t) = \| t\mathbf{x} + \mathbf{y} \|^2 = \langle t\mathbf{x} + \mathbf{y}, t\mathbf{x} + \mathbf{y} \rangle \geq 0, \text{ for all } t \in \mathbb{R},$$

get the Cauchy-Schwarz inequality $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \| \mathbf{x} \| \cdot \| \mathbf{y} \|$. When is the equality obtained?

Exercise 7.2. Recall that we defined a **metric** on $\mathbb{R}^n$ by

$$\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+, \ \rho(\mathbf{x}, \mathbf{y}) = \| \mathbf{x} - \mathbf{y} \|.$$

Consider $n = 2$ and $n = 3$. Sketch the *unit ball of* $\mathbb{R}^n$:

$$B_1(\mathbf{0}) = \{ \mathbf{x} \in \mathbb{R}^n \mid \rho(\mathbf{x}, \mathbf{0}) = \| \mathbf{x} \| < 1 \}.$$

Also sketch the *ball centered at* $\mathbf{x}_0$ *of radius* r:

$$B_r(\mathbf{x}_0) = \{ \mathbf{x} \in \mathbb{R}^n \mid \rho(\mathbf{x}, \mathbf{x}_0) < r \}.$$

Exercise 7.3. Note that one can rephrase the usual definition of limit of a sequence $(a_n)_n$ of real numbers as:

$$(\forall \varepsilon > 0) \ (\exists N = N(\varepsilon)) \text{ s.t. } [\, \rho(a_n, L) < \varepsilon, \ (\forall n \geq N) \,].$$

(a) Give first an intuitive and then a rigorous definition of convergence of a sequence $(\mathbf{x}_n)_n$ in $\mathbb{R}^n$.

(b) Express the convergence that you introduced in terms of convergence of coordinates. (*Hint.* You may find useful the following:

$$\text{if } \mathbf{x} = (x_1, x_2, \ldots, x_n) \text{ then } |x_i| \leq \| \mathbf{x} \|, \forall i = 1, 2, \ldots, n.)$$

Exercise 7.4. Define what it means for a sequence $(\mathbf{x}_n)_n$ in $\mathbb{R}^n$ to be Cauchy. Using the insight that you got in Exercise 7.3(b), show that $\mathbb{R}^n$ is complete.