Exercise 6.1. (a) Given any sequence of real numbers $(a_n)_{n=0}^{\infty}$ one defines two numbers, denoted $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$ and called *limit inferior* and *limit superior*, respectively, as follows:

(1)
$$\liminf_{n\to\infty} a_n = \sup_{n\geq 0}\left(\inf_{k\geq n} a_k\right) = \sup\{\,\inf\{a_k | k\geq n\}\mid n\in\mathbb{N}\}$$
$$= \sup\{b_n\mid n\in\mathbb{N}\},\ \text{ where } b_n = \inf\{a_k | k\geq n\},$$
and
$$\limsup_{n\to\infty} a_n = \inf_{n\geq 0}\left(\sup_{k\geq n} a_k\right).$$

Consider the sequence $(a_n = (-1)^n)_n$. Compute $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$. Do the same for the sequences: $(a_n = 1/n)_n$, $(a_n = n)_n$, $(a_n = 2)_n$.

(b) Show that $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$. When is the equality obtained?

Exercise 6.2. Prove:

Bolzano-Weierstass Theorem. *Every bounded sequence of real numbers has a convergent subsequence.*

Exercise 6.3. Show that every Cauchy sequence is bounded.

Exercise 6.4. Show that if a Cauchy sequence has a convergent subsequence then the entire sequence is convergent (to the same limit).

Exercise 6.5. Prove the following (it is the Completeness Theorem 2.7.4, in our text book):

Cauchy's criterion for convergence. *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*