Exercise 4.2. Show that if a sequence $(a_n)_{n=1}^{\infty}$ converges to L, then any of its subsequences $(a_{n_k})_{n=1}^{\infty}$ converges to the same limit.

Exercise 4.3. If a sequence $(a_n)_{n=1}^{\infty}$ of real numbers converges, then the set $\{a_n \mid n \in \mathbb{N}\}$ is bounded.

5. LUB property

The sequence of statements in the next exercise leads to the Least Upper Bound (LUB) property of $\mathbb{R}$.

Exercise 5.1. Let $A \subset \mathbb{R}$ be non-empty and bounded below.

(a) Show that there exists the biggest *integer* a_0 that is a lower bound for A.

(b) Show that there exists the biggest *rational number* of the form $a_0.a_1$ which is a lower bound for A. Inductively, show that for every $n \in \mathbb{N}^*$ there exists a_n and there esists the biggest rational number of the form $a_0.a_1a_2\dots a_n$ which is a lower bound for A.

(c) Show that

$$\inf A = a_0.a_1a_2\dots a_n \dots .$$

Exercise 5.2. Consider a sequence of nested intervals $I_1 \supset I_2 \supset \cdots \supset I_n \supset \dots$, where $I_n = [x_n, y_n]$. Using the LUB property show that

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

(In words, any intersection of nested closed intervals is non-empty.)

Note. Exercise 5.2 is an important result. It can be used for example to show that closed intervals are *compact* (definition to come). If one were to *define* $\mathbb{R}$ as an **ordered field, containing $\mathbb{Q}$, with the least upper bound property**, then Exercise 5.2 would allow one to prove the *completeness* of $\mathbb{R}$. In the approach taken by our text book, the completeness of $\mathbb{R}$ is already contained in the infinite decimal expansion from the definition of a real number.