Exercise 12.13. Find the discontinuities of $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 0 \quad , & \text{if } x \notin \mathbb{Q}, \\ \dfrac{1}{q} \quad , & \text{if } x = \dfrac{p}{q} \in \mathbb{Q}, \text{ in the lowest terms.} \end{cases}$$

(*Hint*. Consider a sequence $(r_n)_n$ of rational numbers, where $r_n = p_n/q_n$ in the lowest terms, with $q_n \in \mathbb{N}$, which converges to an irrational number x. What can you say about the sequence $(q_n)_n$?)

Exercise 12.14. If f is an increasing function on the interval (a, b), then the one-sided limits of f exist at each point $c \in (a, b)$, and

$$\lim_{x \to c^-} f(x) = L \leq f(c) \leq \lim_{x \to c^+} f(x) = M.$$

Exercise 12.15. Show that a monotone function $f : [a, b] \to \mathbb{R}$ has at most countable many discontinuities.

Exercise 12.16. If $f : [a, b] \to \mathbb{R}$ is monotone and the range of f intersects every non-empty open interval in $[f(a), f(b)]$, then f is continuous.