2. Logic

Exercise 2.1. Check one the following *duality principles*:

$$\neg(p \vee q) \Leftrightarrow (\neg p) \wedge (\neg q) \quad \text{and} \quad \neg(p \wedge q) \Leftrightarrow (\neg p) \vee (\neg q).$$

Exercise 2.2. A **tautology** is a statement that is always true. Show that the following are tautologies:

(a) $p \vee p \leftrightarrow p$ and $p \wedge p \leftrightarrow p$,
(b) $\neg(\neg p) \leftrightarrow p$,
(c) $p \vee \neg p$.

Exercise 2.3. Show one of the following equivalences:

$$\neg[(\forall x)p(x)] \leftrightarrow (\exists x)\neg p(x),$$
$$\neg[(\exists x)p(x)] \leftrightarrow (\forall x)\neg p(x).$$

Exercise 2.4. Consider the proposition

$$p(\varepsilon, \delta) = \text{“ } \delta < \varepsilon \text{ ”, where } \varepsilon, \delta \in \mathbb{R}.$$

Show that

$$\neg[(\forall \varepsilon)(\exists \delta)p(\varepsilon, \delta)] \leftrightarrow (\exists \varepsilon)(\forall \delta)\neg p(\varepsilon, \delta).$$

3. Real numbers

The reference here is **Section 2.2** in our text-book.

Exercise 3.1. The purpose of this exercise is to make you aware of the following characterization:

> *The rational numbers are those real numbers whose decimal expansion is eventually periodic (that is, after a certain rank N, the digits a_n with $n \geq N$ begin to repeat in a certain finite pattern).*

(a) What is the infinite decimal expansion of 1/7? Of 1/11?

(b) We denote an infinitely repeating succession of digits by enclosing the repeating part in parentheses and writing it only once. What rational number has the infinite decimal expansion:

$$0.3(7) = 0.3777777777\ldots?$$

(c) Can you prove the above characterization of rational numbers? Do you see any possible problems or start to feel a bit uneasy with the definition of real numbers via decimal expansion?

Exercise 3.2. This exercise treats a very important concept. Let S be a set. A **(total) order on** S is a relation between the pairs of elements of S, denoted $<$, with the following two properties

(i) if $x, y \in S$ then *one and only one* of the statements

$$x < y, \ x = y, \text{ or } y < x \text{ is true.}$$

(ii) (transitivity) If $x, y, z \in S$, with $x < y$ and $y < z$, then $x < z$.

Construct an order on $\mathbb{R}$. (It is an old friend!) Provide full details, given two distinct real numbers x and y, for the meaning of $x < y$.

Note. By definition, an real number x such that $0 < x$ is called **positive**. After we discuss the operations with real numbers, we shall see that $\mathbb{R}$ is an **ordered field**, and that this order is one of the fundamental characteristics of $\mathbb{R}$.