Group Work Tip #3. Try the following: the complete solution of a problem is written down by **one and only one** member of the group! The others pay attention or offer corrections. Then all move to the next problem and change the writer!

Exercise 8.8. Show that a compact subset of $\mathbb{R}^n$ is closed and bounded.

Exercise 8.9. Show that a closed subset C of a compact metric space X is compact.

Exercise 8.10. Let X be a metric space and $A \subseteq X$. Consider the following conditions:

(K1) every sequence of elements in A has a convergent subsequence with limit in A;
(K2) A is complete and for every $\varepsilon > 0$ there is a finite number $N = N(\varepsilon)$ of points $a_1, a_2, \ldots, a_n$ in A such that $A \subseteq \bigcup_{i=1}^{N} B_\varepsilon(a_i)$.

Show that (K1) implies (K2). (*Hint.* For the covering with balls part, argue by contradiction.)

9. Series

Exercise 9.1. (Test #0 for the convergence of a series.) Show that $\lim_{n\to\infty} a_n = 0$ is necessary for $\sum_{n=1}^{\infty} a_n$ to converge.

Exercise 9.2. Use Cauchy's criterion for convergence of a sequence to get an equivalent statement for the convergence of a series.

Exercise 9.3. (Second part of the **Root test**, the **Test #3** for convergence of a series.) Prove that if $L = \limsup \sqrt[n]{a_n} > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Exercise 9.4. (Fun problem.) Rational powers of real numbers were used in the previous exercise. The purpose of this exercise is to prove the existence of such powers. (Recall that the existence of roots is not contained in any of the axioms of $\mathbb{R}$.)

> **Theorem.** *For every real number $x > 0$ and every positive integer n there is one and only one positive real number y such that $y^n = x$.* (The number y is written $\sqrt[n]{x}$ or $x^{1/n}$.)

The next steps will lead towards a proof of this result.

(a) Consider
$$E = \{\, t \mid t \in \mathbb{R}, t > 0, t^n < x \,\}.$$
Prove that E is non-empty by finding at least an element in it.

(b) Show that $1 + x$ is an upper bound for E, and let $y = \sup E$.

(c) Show that
$$b^n - a^n < (b - a)nb^{n-1}, \text{ when } 0 < a < b.$$

(d) Assume $y^n < x$. Choose h such that $0 < h < 1$ and
$$h < \frac{x - y^n}{n(y + 1)^{n-1}}.$$
Put $a = y$ and $b = y + h$ in the inequality in (c). What can you conclude?

(e) Finish the proof of the Theorem.