Infinite Series and Gambling

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Craps is a popular dice game among those who enjoy gambling. There are many different bets in craps, but we will only focus on two, the Pass Line and the Don't Pass Line. Here is a description of these 2 bets.

Pass Line: Suppose that you bet \$1 on the Pass Line. The shooter (the one who rolls the dice) then rolls for the first time. This is called the "Come Out" roll.

Case 1: If the Come Out roll is either a 7 or an 11, you win your \$1 bet.

Case 2: If the Come Out roll is a 2, 3, or 12, you lose your \$1 bet.

Case 3: Any other number (4,5,6,8,9,10) becomes the "Point". After the Point has been established, the shooter continues rolling until either 7 or the Point is rolled. If the Point occurs first, you win \$1. If the 7 occurs first, you lose your bet.

Don't Pass Line: The Don't Pass Line is almost the opposite of the Pass Line. Suppose that you bet \$1 on the Don't Pass Line.

Case 1: If the Come Out roll is either a 2 or 3, you win your \$1 bet.

Case 2: If the Come Out roll is a 7 or 11, you lose your \$1 bet.

Case 3: If the Come Out roll is a 12, the bet is a push (so you simply get your money back).

Case 4: Any other number (4,5,6,8,9,10) becomes the "Point". After the Point has been established, the shooter continues rolling until either 7 or the Point is rolled. If the 7 occurs first, you win \$1. If the Point occurs first, you lose your bet.

Question: Which is the better bet? That is, on which bet should you expect to lose money more slowly?

Probability of Winning on the Pass Line: There are 36 possible rolls of the dice. The probability that a given number is rolled equals the fraction of the rolls that give this number. For example, the probability that a 4 is rolled equals $\frac{3}{36} = \frac{1}{12}$ (because the possible rolls are $(1,3),(2,2),(3,1)$). One way to win if you bet on the Pass Line is if a 7 or an 11 is the Come Out roll. The probability that a 7 is rolled equals $\frac{6}{36}$ and the probability that an 11 is rolled equals $\frac{2}{36}$. Therefore, the probability that you win in this way equals $\frac{6}{36} + \frac{2}{36} = \frac{8}{36}$.

We now have to determine the probability that you win if a Point is established. Let's work with 4 first. The probability that 4 becomes the Point is $\frac{3}{36}$. Given that 4 becomes the Point, we now need to determine the probability that a 4 is rolled before a 7. We can break up the situations in which a 4 is rolled before a 7 into infinitely many pieces:

(1) A 4 occurs on the first roll.

(2) The first roll is neither a 4 nor a 7, and the second roll is a 4.

(3) The first two rolls are neither 4's nor 7's, and the third roll is a 4.

$\vdots$

(n) The first $n-1$ rolls are neither 4's nor 7's, and the n^{th} roll is a 4 .

$\vdots$

Each of these are mutually exclusive, so to determine the total probability, we determine each individual probability and add them up. The probability that (1) occurs equals $\frac{3}{36}$. The probability that (2) occurs equals $\frac{27}{36} \cdot \frac{3}{36}$ since there are 27 different rolls which are neither a 4 nor a 7, and the outcome of the second roll is completely independent of the outcome of the first roll. Similarly, the probability that (n) occurs equals $(\frac{27}{36})^{n-1} \cdot \frac{3}{36}$. It follows that the probability that you win, given that 4 becomes the Point, equals

$$\begin{aligned}\frac{3}{36} + \frac{27}{36} \cdot \frac{3}{36} + (\frac{27}{36})^2 \cdot \frac{3}{36} + \cdots + (\frac{27}{36})^n \cdot \frac{3}{36} + \ldots &= \sum_{n=0}^{\infty} \frac{3}{36} (\frac{27}{36})^n \\ &= \frac{\frac{3}{36}}{1 - \frac{27}{36}} \\ &= \frac{1}{3}\end{aligned}$$

Therefore, the probability that the 4 becomes the Point and you win equals $\frac{3}{36} \cdot \frac{1}{3} = \frac{1}{36}$.

We've done all of the hard work. The rest are quite similar. In fact, the probability that the Point is 10 and you win also equals $\frac{1}{36}$ because the probability that a 4 is rolled equals the probability that a 10 is rolled. Let's now work on 5 (9 is exactly the same). First, the probability that 5 becomes the Point equals $\frac{4}{36}$. Given that 5 is the point, the probability that you win equals

$$\begin{aligned}\frac{4}{36} + \frac{26}{36} \cdot \frac{4}{36} + (\frac{26}{36})^2 \cdot \frac{4}{36} + \cdots + (\frac{26}{36})^n \cdot \frac{4}{36} + \ldots &= \sum_{n=0}^{\infty} \frac{4}{36} (\frac{26}{36})^n \\ &= \frac{\frac{4}{36}}{1 - \frac{26}{36}} \\ &= \frac{2}{5}\end{aligned}$$

Therefore, the probability that the 5 becomes the Point and you win equals $\frac{4}{36} \cdot \frac{2}{5} = \frac{2}{45}$.

Now we work on 6 (8 is exactly the same). First, the probability that 6 becomes the Point equals $\frac{5}{36}$. Given that 6 is the point, the probability that you win equals

$$\begin{aligned}\frac{5}{36} + \frac{25}{36} \cdot \frac{5}{36} + (\frac{25}{36})^2 \cdot \frac{5}{36} + \cdots + (\frac{25}{36})^n \cdot \frac{5}{36} + \ldots &= \sum_{n=0}^{\infty} \frac{5}{36} (\frac{25}{36})^n \\ &= \frac{\frac{5}{36}}{1 - \frac{25}{36}} \\ &= \frac{5}{11}\end{aligned}$$

Therefore, the probability that the 5 becomes the Point and you win equals $\frac{5}{36} \cdot \frac{5}{11} = \frac{25}{396}$.

We can determine the probability that you win if you bet on the Pass Line by adding up the probability for each of the mutually exclusive possibilities (7 on the Come Out roll, 11 on the Come Out roll, win on a Point of $4, 5, 6, 8, 9, 10$, respectively):

$$\tfrac{6}{36} + \tfrac{2}{36} + \tfrac{1}{36} + \tfrac{2}{45} + \tfrac{25}{396} + \tfrac{25}{396} + \tfrac{2}{45} + \tfrac{1}{36} \quad = \quad \tfrac{244}{495} \quad \approx \quad .492929$$

Probability of Winning on the Don't Pass Line: In order to determine this, you can go through an analysis similar to the one above (and I encourage you to do this if find these calculations interesting). However, there the following approach is much simpler. Notice that there are three mutually exclusive possible outcomes in one "round" of craps. Either the Come Out roll is a 12, or the Pass Line bet wins, or the Don't Pass Line bet wins. The probability that Come out roll is a 12 equals $\frac{1}{36}$. Therefore, if p is the probability that the Don't Pass Line bet wins, then $1 = \frac{1}{36} + \frac{244}{495} + p$. It follows that the probability that a Don't Pass Line bet wins equals:

$$1 - \frac{1}{36} - \frac{244}{495} \quad = \quad \frac{949}{1980} \quad \approx \quad .479293$$

It may now seem that it is wiser to bet on the Pass Line rather than the Don't Pass Line (of course, neither is in your favor). However, we should be a little careful. Failing to win on the Don't Pass Line does not mean that you lose your $1 because the Come Out roll may be a 12. The way to measure which bet is wiser (i.e. which bet you can expect to lose money on more slowly) is to calculate the "Expected Value". This number gives a very accurate measure of what you should expect in the long run (as the number of plays tends to ∞). To determine the Expected Value in this case (it is a bit more complicated in others, but we will get to that later), we take the probability that you win and subtract off the probability that you lose (notice this makes a difference because you can neither win nor lose on a Don't Pass Line bet). We therefore have:

Expected Value of a Pass Line Bet:

$$\frac{244}{495} - \frac{251}{495} \quad = \quad -\frac{7}{495} \quad \approx \quad -.014141$$

Expected Value of a Don't Pass Line Bet:

$$\frac{949}{1980} - \frac{244}{495} \quad = \quad -\frac{3}{220} \quad \approx \quad -.013636$$

This implies that if you always bet on the Pass Line, then as the number of rounds gets extremely large, you can be nearly certain that you will have lost an average of approximately .014141 dollars per round that has been played. Similarly, if you always bet on the Don't Pass Line, then as the number of rounds gets extremely large, you can be nearly certain that you will have lost an average of approximately .013636 dollars per round that has been played. This is stated precisely by "The Law of Large Numbers", which we will eventually prove.

Thus, the Don't Pass Line is the better (i.e. less bad) bet.