Combinations

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Inclusion-Exclusion Principle

Theorem. *Let P be a probability distribution on a sample space Ω, and let $\{A_1,\ A_2,\ \ldots,\ A_n\}$ be a finite set of events. Then*

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i) \; - \sum_{1 \le i < j \le n} P(A_i \cap A_j)$$

$$+ \sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_k) - \cdots .$$

That is, to find the probability that at least one of n events A_i occurs, first add the probability of each event, then subtract the probabilities of all possible two-way intersections, add the probability of all three-way intersections, and so forth.

Hat Check Problem

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- If A_i is the event that the ith element a_i remains fixed under this map, then

$$P(A_i) = \frac{1}{n}.$$

- If we fix a particular pair (a_i, a_j), then

$$P(A_i \bigcap A_j) = \frac{1}{n(n-1)}.$$

- The number of terms of the form $P(A_i \bigcap A_j)$ is $\left(\begin{array}{c} n \\ 2 \end{array} \right)$.

- For any three events A_1, A_2, A_3

$$P(A_i \cap A_j \cap A_k) = \frac{(n-3)!}{n!} = \frac{1}{n(n-1)(n-2)} ,$$

and the number of such terms is

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{3!} .$$

- Hence

$$P(\text{at least one fixed point}) = 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots (-1)^{n-1}\frac{1}{n!}$$

and

$$P(\text{no fixed point}) = \frac{1}{2!} - \frac{1}{3!} + \cdots (-1)^n \frac{1}{n!} \; .$$

n	Probability that no one gets his own hat back
3	.333333
4	.375
5	.366667
6	.368056
7	.367857
8	.367882
9	.367879
10	.367879

Problems

Show that the number of ways that one can put n different objects into three boxes with a in the first, b in the second, and c in the third is $n!/(a!\,b!\,c!)$.

Suppose that a die is rolled 20 independent times, and each time we record whether or not the event $\{2, 3, 5, 6\}$ has occurred.

1. What is the distribution of the number of times this event occurs in 20 rolls?

2. Calculate the probability that the event occurs five times.

Suppose that a basketball player sinks a basket from a certain position on the court with probability 0.35.

1. What is the probability that the player sinks three baskets in ten independent throws?

2. What is the probability that the player throws ten times before obtaining the first basket?

3. What is the probability that the player throws ten times before obtaining two baskets?

Poker Hands

Suppose that we have a standard 52 card deck.

- In a poker game does a *straight* beat *three of a kind*? (straight: five cards in a sequence regardless of suit, but not a royal or a straight flush). Why?
- Does a *straight* beat a *full house*? Why?
- Why does a *four of a kind* beat a *full house?*

Show that

$$b(n, p, j) = \frac{p}{q} \left(\frac{n - j + 1}{j} \right) b(n, p, j - 1) \ ,$$

for $j \geq 1$. Use this fact to determine the value or values of j which give $b(n, p, j)$ its greatest value.

Conditional Probability

- Suppose that we draw two cards successively without replacement from a standard deck D.

- Consider the event $A = \{\text{the second card is a king}\}$. What is $P(A)$?

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- Suppose that we draw two cards successively without replacement from a standard deck D.

- Consider the event $A = \{\text{the second card is a king}\}$. What is $P(A)$?

- Suppose that you are told after the first card is drawn that is was a king. What is the probability $P(A|B)$ that the second card is a king?

Definition

Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_r\}$ be the original sample space with distribution function $m(\omega_j)$ assigned. Suppose we learn that the event E has occurred.

- If a sample point ω_j is not in E, we want $m(\omega_j|E) = 0$.

- For ω_k in E, we should have the same relative magnitudes that they had before we learned that E had occurred:

$$m(\omega_k|E) = cm(\omega_k).$$

But we must also have

$$\sum_E m(\omega_k|E) = c \sum_E m(\omega_k) = 1 \ .$$

Thus,

$$c = \frac{1}{\sum_E m(\omega_k)} = \frac{1}{P(E)} \ .$$

Definition. *The* conditional distribution *given* E *is the distribution on* Ω *defined by*

$$m(\omega_k|E) = \frac{m(\omega_k)}{P(E)}$$

for ω_k *in* E, *and* $m(\omega_k|E) = 0$ *for* ω *not in* E.

Then, for a general event F,

$$P(F|E) = \sum_{F \cap E} m(\omega_k|E) = \sum_{F \cap E} \frac{m(\omega_k)}{P(E)} = \frac{P(F \cap E)}{P(E)} \ .$$

We call $P(F|E)$ the *conditional probability of* F *occurring given that* E *occurs.*