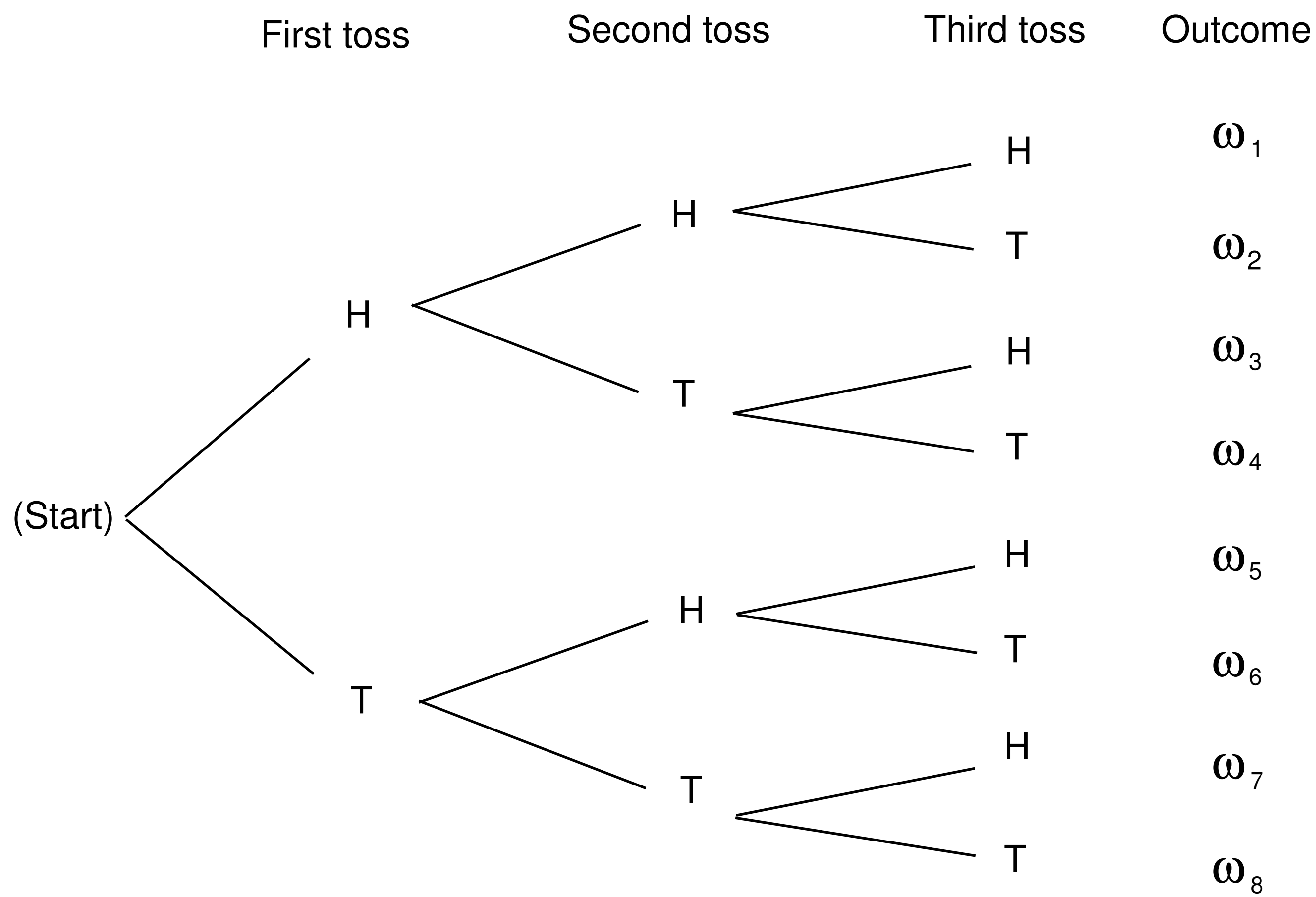
Combinatorics: Permutations

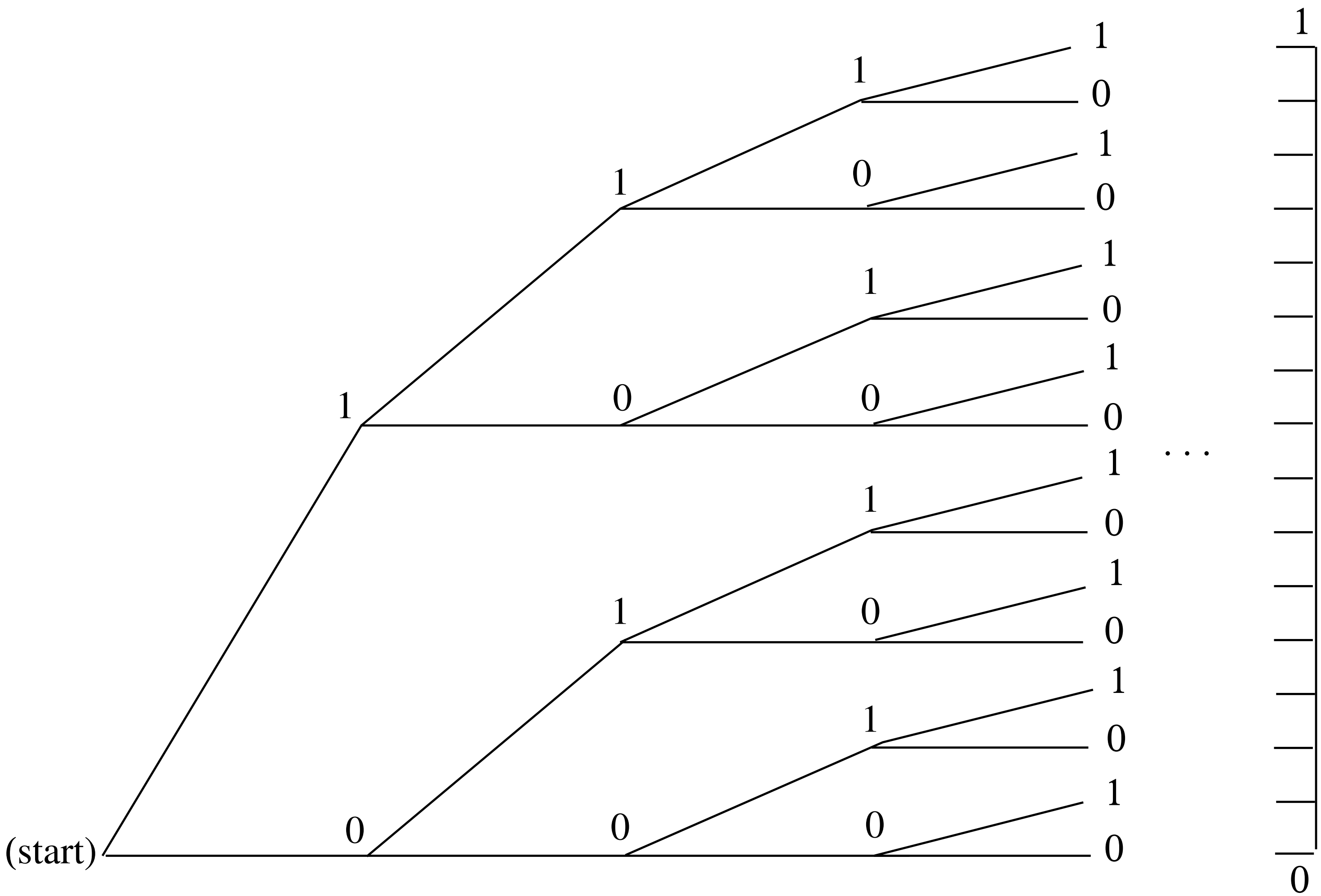
April 10, 2006

Finite and Infinite Tree

- Suppose we toss a coin three times.
- How can we describe the sample space and the distribution function of this variable?


First toss
Second toss
Third toss
Outcome
(Start)
H
T
H
T
H
T
H
T
H
T
H
T
H
T
ω_1
ω_2
ω_3
ω_4
ω_5
ω_6
ω_7
ω_8

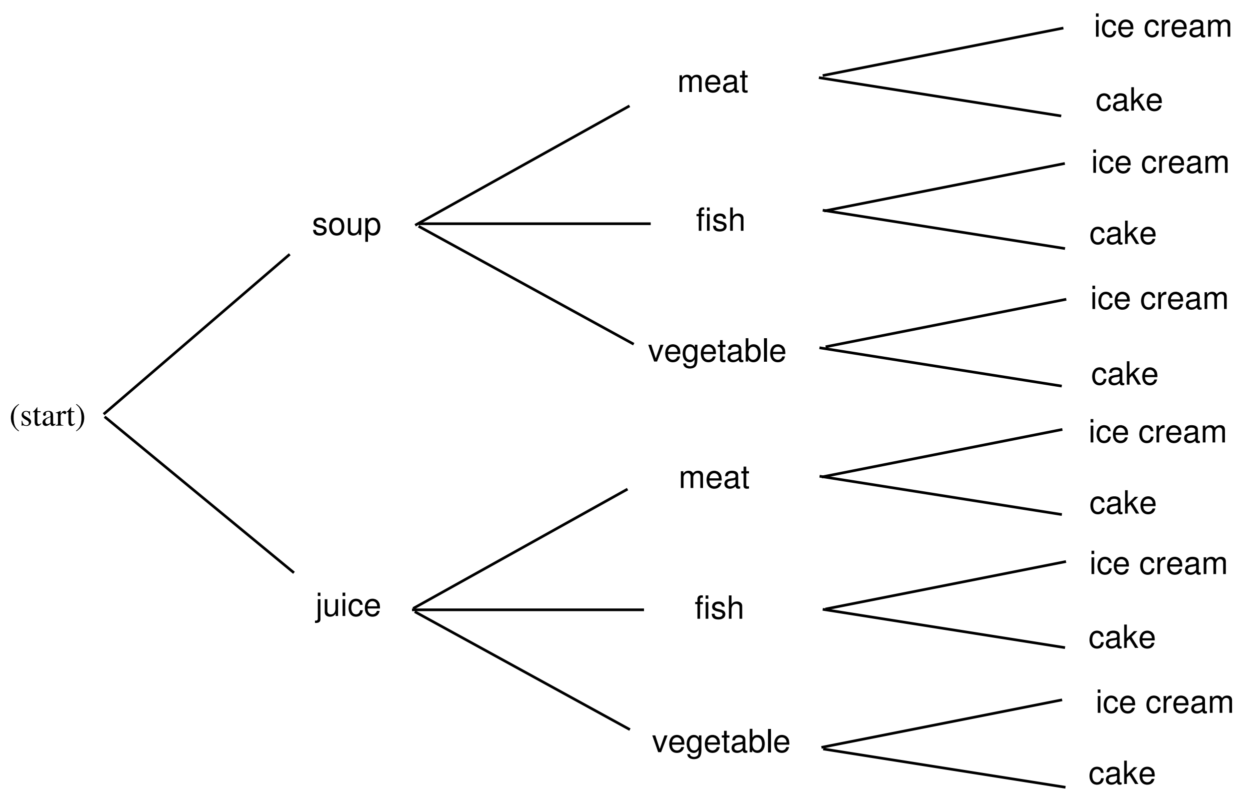
- Suppose now the a fair coin is tossed repeatedly, without stopping.
- We expect that the natural sample space is a binary tree with an infinite number of stages.


(start)
1
0
1
0
1
0
1
0
1
0
1
0
1
0
1
0
1
0
1
0
1
0
1
0
1
0
1
0
. . .
1
0

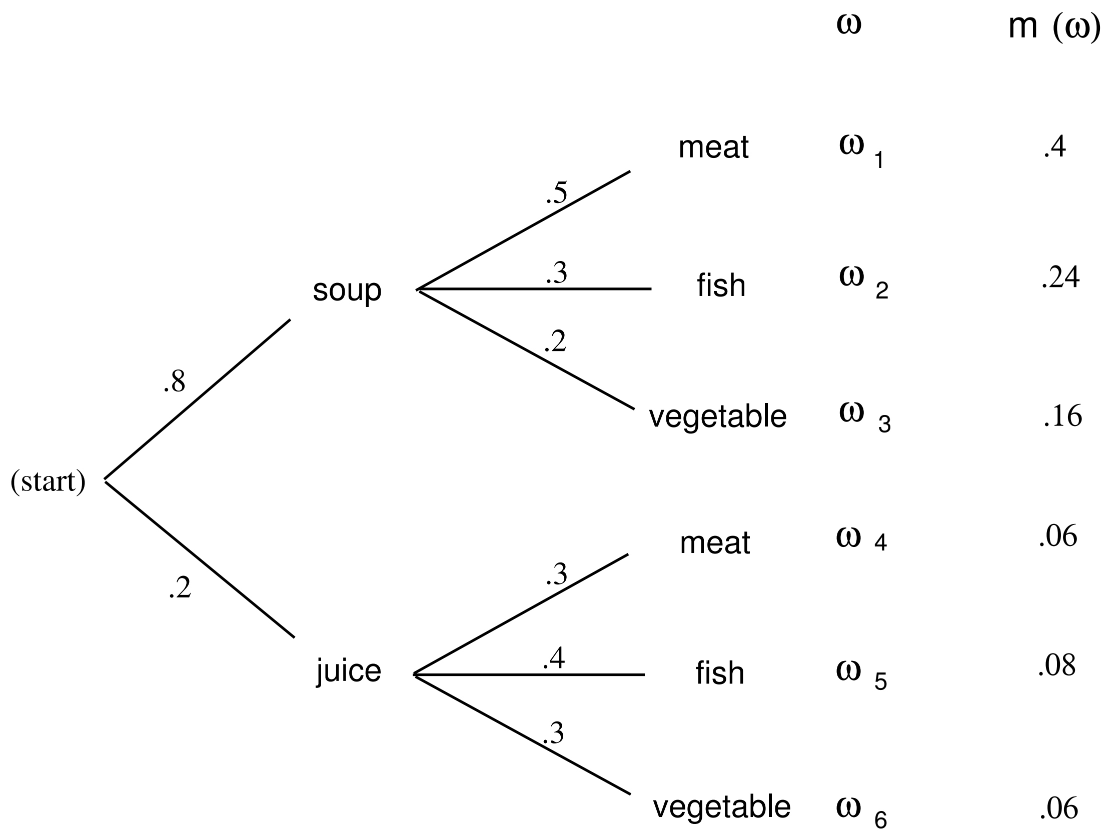
Example

You are eating at Émile's restaurant and the waiter informs you that you have

1. two choices for appetizers: soup or juice;
2. three for the main course: a meat, fish, or vegetable dish; and
3. two for dessert: ice cream or cake. How many possible choices do you have for your complete meal?


(start)
soup
juice
meat
fish
vegetable
meat
fish
vegetable
ice cream
cake
ice cream
cake
ice cream
cake
ice cream
cake
ice cream
cake
ice cream
cake

Assume that the owner of Émile's restaurant has observed that 80 percent of his customers choose the soup for an appetizer and 20 percent choose juice. Of those who choose soup, 50 percent choose meat, 30 percent choose fish, and 20 percent choose the vegetable dish. Of those who choose juice for an appetizer, 30 percent choose meat, 40 percent choose fish, and 30 percent choose the vegetable dish. What is the probability that a customer eats vegetable.


ω
m (ω)
meat
ω 1
.4
.5
soup
.3
fish
ω 2
.24
.2
.8
vegetable
ω 3
.16
(start)
meat
ω 4
.06
.2
.3
juice
.4
fish
ω 5
.08
.3
vegetable
ω 6
.06

Prove that at least two people in Atlanta, Georgia, have the same initials, assuming no one has more than four initials.

Birthday Problem

How many people do we need to have in a room to make it a favorable bet (probability of success greater than $1/2$) that two people in the room will have the same birthday?

License Plates

In Clark County, Nevada license plates have six symbols, three leters folowed by thre numbers. The first letter on the plate is either a T or a C . How many license plates of this type are available?

As I was going to St. Ives,
I met a man with seven wives,
Each wife had seven sacks,
Each sack had seven cats,
Each cat had seven kits.
Kits, cats, sacks and wives,
How many were going to St. Ives?

Permutations

Definition. *Let A be any finite set. A* permutation *of A is a one-to-one mapping of A onto itself.*

$$\sigma = \begin{pmatrix} a & b & c & d \\ b & d & c & a \end{pmatrix}.$$

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Theorem. *The total number of permutations of a set A of n elements is given by $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$.*

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Definition. *Let A be an n-element set, and let k be an integer between 0 and n. Then a k-permutation of A is an ordered listing of a subset of A of size k.*

Theorem. *The total number of k-permutations of a set A of n elements is given by $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1)$.*

Factorials

Definition. *The number* $n \cdot (n-1) \cdot \ldots \cdot 1$ *is called* n factorial.

n	$n!$
0	1
1	1
2	2
3	6
4	24
5	120
6	720
7	5040
8	40320
9	362880
10	3628800

Stirling's Formula

Definition. *Let a_n and b_n be two sequences of numbers. We say that a_n is* asymptotically equal *to b_n, and write $a_n \sim b_n$, if*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 \ .$$

Theorem. *The sequence $n!$ is asymptotically equal to*

$$n^n e^{-n} \sqrt{2\pi n} \ .$$