Continuous Density Functions

April 7, 2006

Continuous Space Coordinates

- We suppose that suitable coordinates can be introduced into the sample space Ω.
- We can regard Ω as a subset of $\mathbf{R}^n$.
- We call such a sample space a *continuous sample space*.
- Let X be a random variable which represents the outcome of the experiment.
- Such a random variable is called a *continuous random variable*.

Density Functions and Cumulative Distribution Function

- Let X be a continuous real-valued random variable.
- A density function for X is a real-valued function f which satisfies

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

  for all $a,\ b \in \mathbb{R}$.

- The cumulative distribution function of X is defined by the equation

$$F_X(x) = P(X \leq x)\ .$$

Examples

- In the dart game what is the distribution of the distance of the dart from the center of the target?

- What is its density?

Examples

- In the dart game what is the distribution of the distance of the dart from the center of the target?
- What is its density?
- The sample space Ω is the unit disk with coordinates (X, Y).
- Let $Z = \sqrt{X^2 + Y^2}$ represent the distance from the center of the target.

- Let E be the event $\{Z \leq z\}$.

- The distribution function F_Z of Z is given by

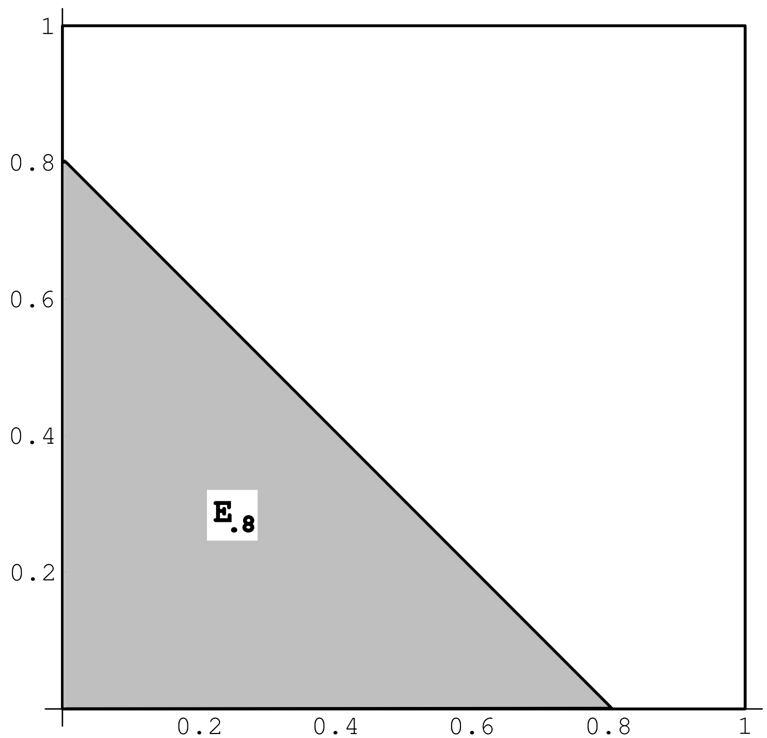
$$\begin{aligned} F_Z(z) &= P(Z \leq z) \\ &= \frac{\text{Area of } E}{\text{Area of target}} \, . \end{aligned}$$

- A real number is chosen at random from $[0, 1]$ with uniform probability, and then this number is squared.
- Let X represent the result.
- What is the cumulative distribution function of X?
- What is the density of X?

- Consider a random variable defined to be the sum of two random real numbers chosen uniformly from $[0, 1]$.
- What is the cumulative distribution function of X?
- What is the density of X?

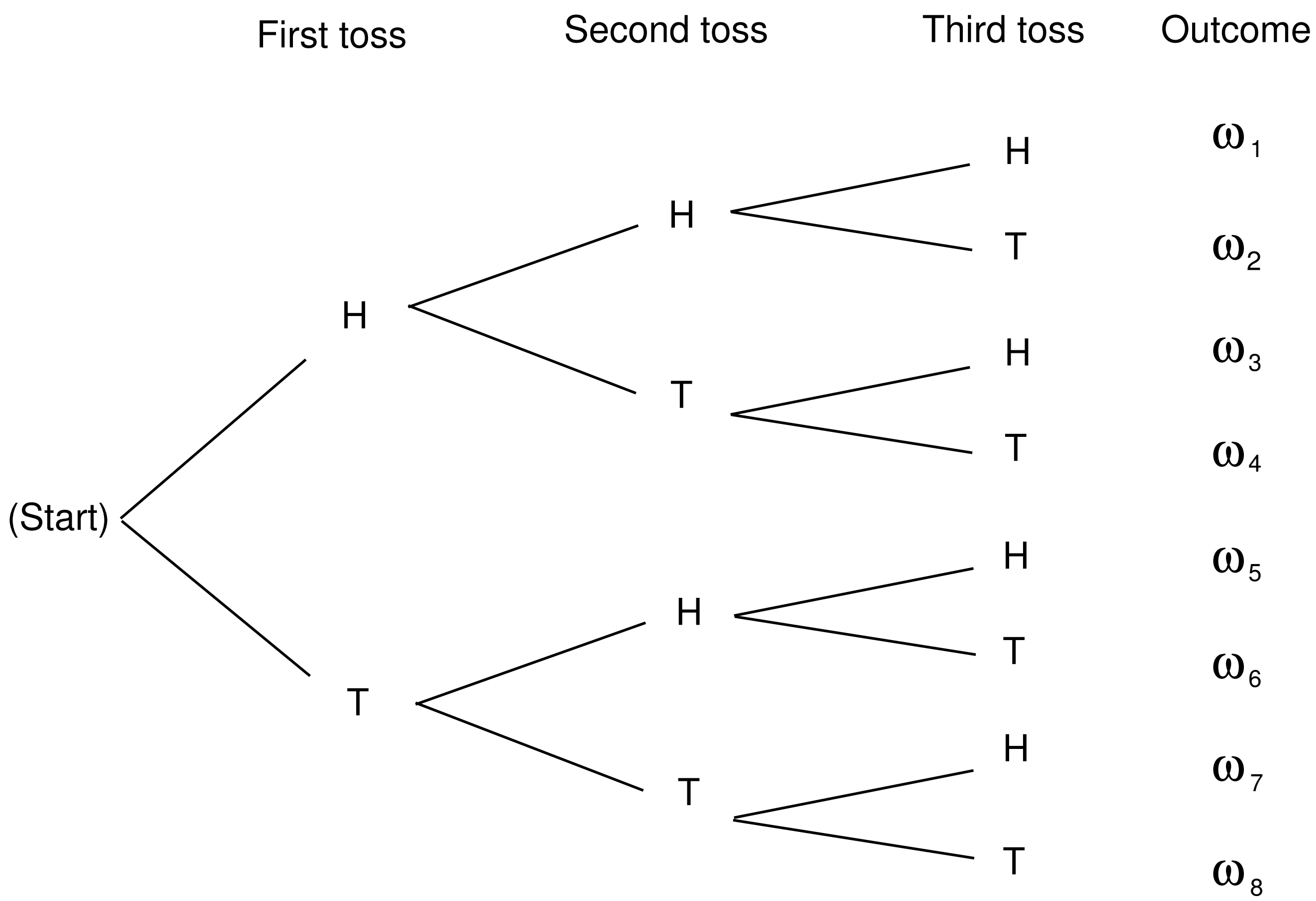
- The sample space Ω isthe unit square with uniform density.

- Let E_z denote the event that $Z \leq z$.

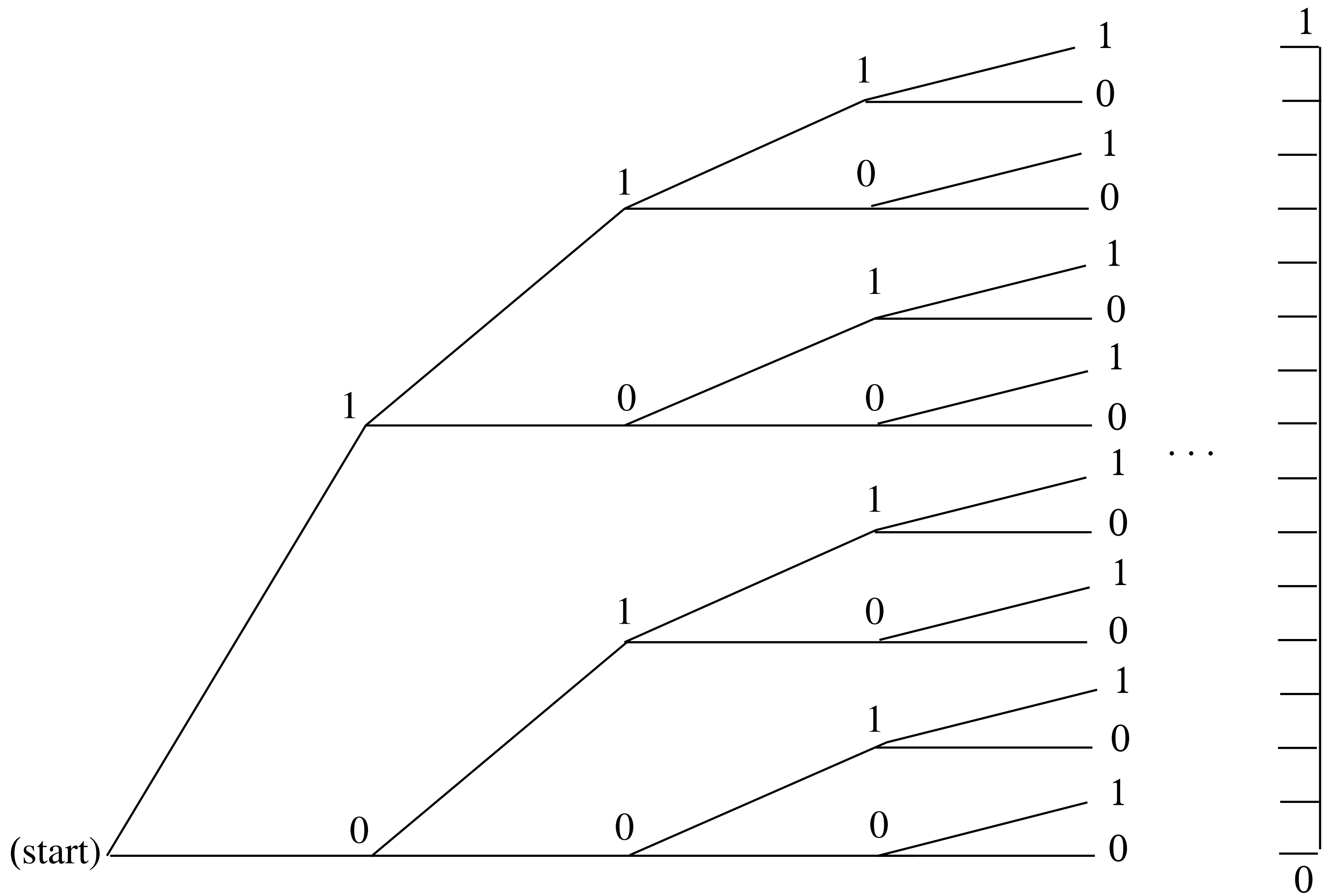


Finite and Infinite Tree

- Suppose we toss a coin three times.
- How can we describe the sample space and the distribution function of this variable?


First toss
Second toss
Third toss
Outcome
(Start)
H
T
H
T
H
T
H
T
H
T
H
T
H
T
ω_1
ω_2
ω_3
ω_4
ω_5
ω_6
ω_7
ω_8

- Suppose now the a fair coin is tossed repeatedly, without stopping.
- We expect that the natural sample space is a binary tree with an infinite number of stages.


(start)
1
0
1
0
1
0
1
0
1
0
1
0
1
0
1
0
1
0
1
0
1
0
1
0
1
0
1
0
. . .
1
0

The exponential density

- We observe a sequence of occurrences which occur at "random" times.

- Define a random variable X to denote the time between successive occurrences.

- Often we can model X by using the exponential density:

$$f(t) = \begin{cases} \lambda e^{-\lambda t}, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$