Continuous Density Functions

April 5, 2006

Bertran's Paradox (cont'd)

- A chord of a circle is a line segment both of whose endpoints lie on the circle.
- Suppose that a chord is drawn *at random* in a unit circle.
- What is the probability that its length exceeds $\sqrt{3}$?

Spinners (revisited)

- A spinner consists of a circle of unit circumference and a pointer.
- Let's simulate this experiment such that we produce a graph bar with the property that on each interval, the *area*, rather than the height, of the bar is equal to the fraction of outcomes that fell in the corresponding interval.
- Use the program *Areabargraph*.

- We would like
$$P(c \le X < d) = d - c.$$

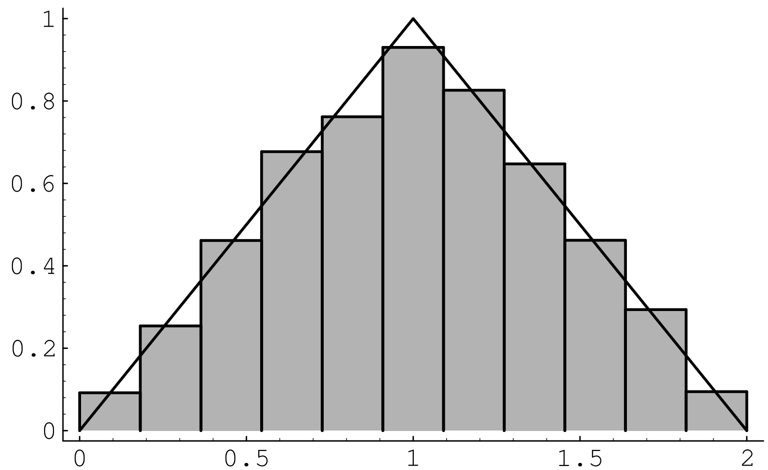
- If we let $E = [c, d]$, then we can write the above formula in the form
$$P(E) = \int_E f(x)\,dx\ ,$$
where $f(x)$ is the constant function with value 1.

- The function $f(x)$ is called the *density function* of the random variable X.

Sum of random numbers

- Choose two random real numbers in $[0, 1]$ and add them together.
- Let X be the sum.
- How is X distributed?

Sum of random numbers ...



- It appears that the function defined by

$$f(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1, \\ 2 - x, & \text{if } 1 < x \leq 2 \end{cases}$$

fits the data very well.

- Suppose that we choose 100 random numbers in $[0, 1]$, and let X represent their sum.

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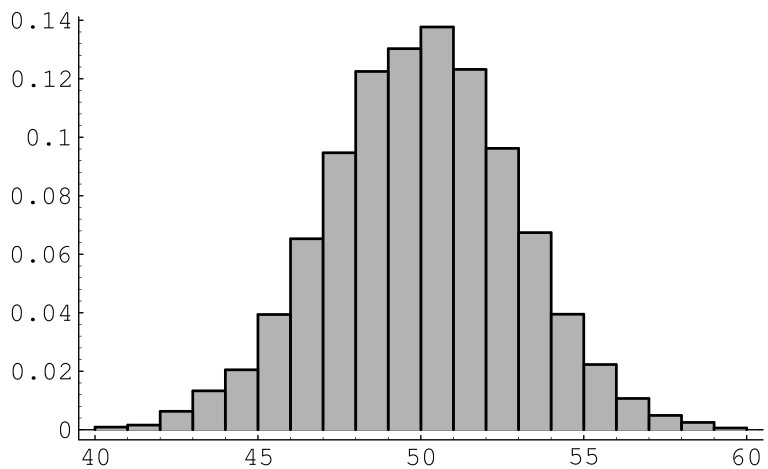

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$$P(E) = \frac{\text{area of } E}{\text{area of target}} = \frac{\text{area of } E}{\pi} \ .$$

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- if $E = \{\,(x, y) : x^2 + y^2 \le a^2\,\}$ is the event that the dart lands within distance $a < 1$ of the center of the target, then

$$P(E) = \frac{\pi a^2}{\pi} = a^2 \ .$$

Sample Space Coordinates

- A sample space Ω which is a subset of $\mathbf{R}^n$ is called a *continuous sample space*.

- Let X be a random variable which represents the outcome of the experiment.

- Such a random variable is called a *continuous random variable*.

Density Functions of Continuous Random Variables

- Let X be a continuous real-valued random variable. A density function for X is a real-valued function f which satisfies

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

  for all $a,\ b \in \mathbb{R}$.

- if E is a subset of $\mathbb{R}$, then

$$P(X \in E) = \int_E f(x)\,dx\ .$$

Examples

- **The spinner**: $\Omega = [0,1]$ and

$$f(x) = \begin{cases} 1, & \text{if } 0 \le x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

- **The dart game**: $\Omega = \{(x,y)\ :\ x^2 + y^2 \le 1\}$, and

$$f(x,y) = \begin{cases} 1/\pi, & \text{if } x^2 + y^2 \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Cumulative Distribution Functions of Continuous Random Variables

- Let X be a continuous real-valued random variable.
- Then the *cumulative distribution* function of X is defined by the equation

$$F_X(x) = P(X \leq x) \ .$$

Theorem. *Let* X *be a continuous real-valued random variable with density function* $f(x)$*. Then the function defined by*

$$F(x) = \int_{-\infty}^{x} f(t)\, dt$$

is the cumulative distribution function of X*. Furthermore, we have*

$$\frac{d}{dx} F(x) = f(x)\ .$$