Random Walks in Euclidian Space

05/19/2006

Definition. *1. Let* $\{X_k\}_{k=1}^{\infty}$ *be a sequence of independent, identically distributed discrete random variables. For each positive integer* n*, we let* S_n *denote the sum* $X_1 + X_2 + \cdots + X_n$*. The sequence* $\{S_n\}_{n=1}^{\infty}$ *is called a* random walk.

2. We say that an equalization *has occurred, or there is a* return to the origin *at time* n*, if* $S_n = 0$

Theorem. *The probability of a return to the origin at time $2m$ is given by*

$$u_{2m} = \binom{2m}{m} 2^{-2m} \ .$$

Theorem. *For $m \geq 1$, the probability of a first return to the origin at time $2m$ is given by*

$$f_{2m} = \frac{u_{2m}}{2m-1} = \frac{\binom{2m}{m}}{(2m-1)2^{2m}} .$$

- Generating functions

$$U(x) = \sum_{m=0}^{\infty} u_{2m} x^m$$

and

$$F(x) = \sum_{m=0}^{\infty} f_{2m} x^m .$$

Probability of Eventual Return

- In the symmetric random walk process in $\mathbf{R}^m$, what is the probability that the particle eventually returns to the origin?

Eventual Return in $\mathbf{R}^1$

- We will define w_n to be the probability that a first return has occurred no later than time n.

- Define the probability that the particle eventually returns to the origin to be

$$w_* = \lim_{n \to \infty} w_n \ .$$

- In terms of the f_n probabilities, we see that

$$w_{2n} = \sum_{i=1}^{n} f_{2i} \ .$$

Theorem. *With probability one, the particle returns to the origin.*

Eventual Return in $\mathbf{R}^m$

- We define $f_{2n}^{(m)}$ to be the probability that the first return to the origin in $\mathbf{R}^m$ occurs at time $2n$.

- The quantity $u_{2n}^{(m)}$ is defined in a similar manner.

- For all $m \geq 1$,

$$u_{2n}^{(m)} = f_0^{(m)} u_{2n}^{(m)} + f_2^{(m)} u_{2n-2}^{(m)} + \cdots + f_{2n}^{(m)} u_0^{(m)} \ .$$

- Define

$$U^{(m)}(x) = \sum_{n=0}^{\infty} u_{2n}^{(m)} x^n$$

and

$$F^{(m)}(x) = \sum_{n=0}^{\infty} f_{2n}^{(m)} x^n \ .$$

-

$$w_*^{(m)} = \lim_{x \uparrow 1} F^{(m)}(x) = \lim_{x \uparrow 1} \frac{U^{(m)}(x) - 1}{U^{(m)}(x)} \ ,$$

- In $\mathbb{R}^2$ the probability of eventual return is 1.

- In $\mathbb{R}^3$ the probability of eventual return is *strictly* less than 1.

Expected Number of Equalizations

- Derive a formula for the expected number of equalizations in a random walk of length $2m$.

Expected Number of Equalizations

- Derive a formula for the expected number of equalizations in a random walk of length $2m$.

- Define g_{2m} to be the number of equalizations among all of the random walks of length $2m$.

- We define $g_0 = 0$.

- We define the generating function $G(x)$:

$$G(x) = \sum_{k=0}^{\infty} g_{2k} x^k \ .$$

- We consider m to be a fixed positive integer, and consider the set of all paths of length $2m$ as the disjoint union

$$E_2 \cup E_4 \cup \cdots \cup E_{2m} \cup H \ ,$$

- We claim that the number of equalizations among all paths belonging to the set E_{2k} is equal to

$$|E_{2k}| + 2^{2k} f_{2k} g_{2m-2k} \ .$$

- The functional equation is

$$G(x) = F(4x)G(x) + \frac{1}{1-4x} - U(4x) \ .$$

- If we simplify, we obtain

$$G(x) = \frac{1}{(1-4x)^{3/2}} - \frac{1}{(1-4x)} \ .$$

- The expected number of equalizations among all paths of length $2m$ is

$$\frac{g_{2m}}{2^{2m}} \sim \sqrt{\frac{2}{\pi}}\sqrt{2m}\ .$$