Facts about the Fourier transforms

Recall the Fourier transform is given by,

$$\hat{f}(t) = (f(x))\hat{} = \int_{-\infty}^{\infty} e^{ixt} f(x)dx$$

and the inverse Fourier transform is given by

$$\check{f}(x) = (f(t))\check{} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} f(t)dt.$$

The Fourier transform can be applied to a wide range of functions (as well as measures), but if you want to be safe it nice to apply it to functions f with the property that $\int_{-\infty}^{\infty} |f|^2 dx < \infty$. In fact if you take the vector space of "all" such functions, then the Fourier transform "is" an invertible isomorphism of this vector space of functions onto itself. In particular the Fourier transform has no problems with bounded pdfs. First I will collect the facts about the Fourier transform derived in lecture on 5/17/02.

1. Linearity
$$(af(x) + g(x))\hat{} = a\hat{f}(t) + \hat{g}(t)$$

2. Differentiation to polynomial multiplication
$$\left(\frac{df}{dx}(x)\right)\hat{} = -it\hat{f}(t)$$

3. Scaling for $a > 0$
$$(\sqrt{a}f(ax))\hat{} = \frac{1}{\sqrt{a}}\hat{f}\left(\frac{t}{a}\right)$$

4. Translation to phase-shift
$$(f(x+a))\hat{} = e^{-iat}\hat{f}(t)$$

5. Phase-shift to translation
$$(e^{iat}f(x))\hat{} = \hat{f}(t+a)$$

6. Convolution to product
$$(f \star g)\hat{}(t) = \hat{f}(t)\hat{g}(t)$$

7. Self adjointedness

$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)dx = \int_{-\infty}^{\infty} \hat{f}(x)g(x)dx$$

8. An Eigen-vector

$$\left(e^{-\frac{x^2}{2}}\right)\hat{} = \sqrt{2\pi}e^{-\frac{t^2}{2}}$$

9. Gaussian pre-image

$$\left(\frac{e^{-i\mu x}e^{-\frac{\sigma^2 x^2}{2}}}{2\pi}\right)\hat{} = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(t-\mu)^2}{2\sigma^2}}$$

10. The identity transformation in disguise

$$\lim_{\sigma\to 0}\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-t)^2}{2\sigma^2}} f(x)dx = f(t)$$

11. Inversion

$$\check{\hat{f}}(x) = f(x)$$

Recall the first 8 facts were straight forward and independent calculations while 9 uses facts 3,5, and 8, and fact 11 relies on 7,9, and 10.

In our second lecture we applied the Fouyrrier transform to discover the distibution which satisfiy $\frac{S_n}{n^\alpha} = X$. After this I messed up the below computation of $(f_{S_n^\star})\hat{}$. I hope no one hold this againdst the beautiful Fourier transfom! next time we'll get a chance to whatch this compuatation in action.

$$\begin{aligned}
f_{S_n^\star}(x) &= f_{\sum_{i=1}^n \frac{X_i-\mu}{\sigma\sqrt{n}}}(x) \\
&= f_{\frac{1}{\sigma\sqrt{n}}(\sum_{i=1}^n X_i - n\mu)}(x) \\
&= \sigma\sqrt{n} f_{S_n}(\sigma\sqrt{n}x + n\mu) \\
&= \sigma\sqrt{n} f_{S_n}\left(\sigma\sqrt{n}\left(x + \frac{\sqrt{n}\mu}{\sigma}\right)\right)
\end{aligned}$$

So we have that

$$(f_{S_n^\star})\hat{}(t) = e^{-i\frac{\sqrt{n}\mu}{\sigma}}\sigma\sqrt{n}\left(f_{S_n}(\sigma\sqrt{n}x)\right)\hat{}(t)$$

$$= e^{-i\frac{\sqrt{n}\mu}{\sigma}} \hat{f}_{S_n}\left(\frac{t}{\sigma\sqrt{n}}\right)$$
$$= e^{-i\frac{\sqrt{n}\mu}{\sigma}} \left(\hat{f}_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n$$