Take Home Exam

Hints: $\int_a^b f(x)dx$ can be approximated with maple for any numbers a and b (or even with b=infinity) via *evalf(int(f(x),x=a..b));* . Also it may be useful to recall that $\sum_{i=0}^{N} r^i = \frac{1-r^{N+1}}{1-r}$.

1. (**Possible helpfull clarification:** This problem will requires no simulation, however you can use maple to help you evaluate any difficult looking integrals or sums). Imagine that you have saved some baby sitting money to go on a fishing trip with your uncle Roy. The only requirement is that your fishing line be at least 50 meters long. You can't go to the fishing supply store because you are busing baby sitting your niece Emily. Fortunately in your basement your roommate stores her fishing line making machine. You know the machine starts making a piece of fishing line with an **attaching device** on the initial end at a rate of 1 meter per minute (see the figure). The machine has a failure rate that is exponentially distributed with $\lambda = 0.013$. Your roommate is out of town and you don't know how to reset the machine once it fails, so you must hope that the machine produces the needed 50 meters of fishing line before failing.

   (a) What is the probability that the machine makes enough line for you to go on the trip?

   (b) Suppose the machine has produced its line, which is a big pile on the floor. Oh no, Emily found a pair of scissors and cut the line! Assuming that Emily's cut was made uniform randomly, what is the probability that the half of the line with the attaching device on it is still long enoughto allow you to go on the trip? (**Possible helpful clarification**: This problem assumes that you do not know the length of the line on floor before Emily makes her cut. Imagine that you let the machine run then came back much later and noticed that it had indeed stopped, leaving you with a big pile of line on the ground. Potentially this line is longer than 50 meters, potentially it is smaller, and Emily cuts this line before you have a chance to measure it.)

2. (This problem is an elaboration of Problems 9 and 10 p. 72.) Recall we learned to simulate how long we expect to "wait for a bus" given the pdf of the initial waiting (as in exponential and the uniform on [5,15] distributions in 9 and 10). In this problem you may compute the

expected value of a random variable X determined by a pdf $f(t)$ via $E(X) = \int_{-\infty}^{\infty} tf(t)dt$ (**Possible helpful clarification**: notice that a distribution function is defined everywhere, but may be zero over large regions like (-infinty,0) in this problem.).

(a) Construct a pdf for the initial wait with expected value 10 minutes that has the property that the initial waiting time can **never** be within a minute of its expect waiting time of ten minutes. (**Possible helpful clarification**: Indeed, it would be recommended that you use a continuous distribution since the formulae are given in terms of integral and that you are modeling the arrival time of a bus).

(b) Using your density function, simulate the expect waiting time if you arrived at time 100.

(c) In class we deduced that the expected waiting time when we arrive at time 100 should be about $W = 1/20 \int_{-\infty}^{\infty} t^2 f(t)dt$. Articulate how likely this integral's value is to be the actual expected waiting time given your empirically determined estimate of the expected waiting time from part 2b. (**Possible helpful clarification**: There is a typo here namely we need 1/20 not 1/10 as stated. Notice that this problem is asking you to test the hypothesis that the integral produces the correct value.)

3. First reread the Watson and Holmes problem, problem 22 p. 91. Now Holmes says, "Watson I think you misunderstand watch counterfeiting. In such a circumstance, the watches are **released** into the community as they are produced. Let us call the time it takes to produce a watch ΔT. Suppose a counterfeiter releases a watch into the community at time S. Let A be the event that the watch is **recovered** in the time interval $[T, T+\Delta T]$ with $T \geq S$ and let B be the event that the watch was not recovered in the time interval $[S, T]$, then we may assume $P(A|B) = p$ for some p." (**Possible helpful clarification**: I have changed the strict inequalities to inequalities, this makes no actual difference since we are veiwing any partiular time (measured with an infinite precision clock) as having zero probability of occurring, but a student found the original choices confusing. I hope these choices are less confusing.)

(a) Under Holmes' assumption, suppose a watch is released at time S. For each $n \geq 0$ let C_n be the event that the watch is recovered in the time interval $[S + n\Delta T, S + (n+1)\Delta T]$. Compute $P(C_n)$.

(b) Suppose the counterfeiter releases his first watch at time 0, and releases a single watch at each subsequent time $j\Delta T$. Call the watch released at time $j\Delta T$ the *jth* watch. Let the random variable X_j be 1 if the jth watch has been recovered and 0 otherwise. Suppose the time now is $N\Delta T$, what is the expected value of X_j? (Hint: X_j is an indicator function.)

(c) Notice $R = \sum_{i=0}^{N-1} X_i$ is the number of the N released watches recovered at time $N\Delta T$. Use the first fundamental mystery of probability to compute $E(R)$ in terms of q and N, and then argue based on this computation that $q = 1 - p \approx 1 - \frac{1}{N-E(R)}$. (**Possible helpful clarification:** i in the X_i was previously a j, which was a typo).

(d) Now we'd like to find the most likely N given our data, so we introduce the random variable Y_j which is j if the jth watch has been recovered and 0 otherwise, and we look at the random variable $M = max(Y_0, \ldots, Y_{N-1})$. Given the description of the situation in problem 22 p. 91 what are good estimates $E(R)$ and $E(M)$.

(e) Argue that in order to find the most likely N that it is reasonable to find the N which most closely produces your proposed $E(M)$.

(f) By simulating find a good value of N, and answer whose estimate is better now, Watson's or Holmes'?