Mathematics 5
Winter Term 2008
The World According to Mathematics

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<u>Class Discussion: Week #9</u>

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Today we are going to conduct a thought experiment involving objects moving at speeds close to the speed of light. We will begin with an excerpt from *Mathematics and Knowledge: Models of Reality,* Section 9.10, a dialogue between Yma and her mother:

Yma: What is mass?

Mom: Mass is a measure of the substance, the *stuff*, of the rock.

Yma: And the speed of light?

Mom: The speed of light is $3x10^8$ meters per second, or 186, 000 miles per second.

Yma: Hmm. That's pretty fast. I have heard that nothing can go faster than the speed of light. That's true, isn't it?

Mom: Yes, it certainly is. One of Einstein's assumptions in his theory of special relativity is that the speed of light is always the same, no matter the velocity of the source or the recipient.

Yma: Can you elaborate on that a bit?

Mom: Certainly. Suppose a train is moving at 80 miles per hour, and someone on the train throws a ball at a speed of 10 miles an hour to a friend several rows ahead. If you are standing on the platform watching the train go by, then what do you think you see as the speed of the ball?

Yma: Well, I guess I would see it as the sum of the two speeds: 80 + 10 = 90 miles per hour. Right?

Mom: Yes, in other words, from your perspective on the platform, you have to add the two speeds to get the speed of the ball.

Yma: Right.

Mom: Now, suppose the train is going at, say, three-fourths the speed of light and someone shines a flashlight (light beam traveling at speed of light) toward the front of the train. What is the speed of the light that you observe from the platform?

Yma: Hmm... I see that the speed of light is different. I can't solve the problem the same way.

Mom: Why?

Yma: Well, you said that the speed of light is always the same, namely, c, no matter the velocity of the source (here the train moving at $3/4c$) or of the recipient (here the person on the platform). So, the answer must be c. Right?

Mom: Right you are. So, here you don't just add the speeds to get $1\,3/4c$.

Yma: I knew that that was wrong anyway because $1\,3/4c$ is bigger than c, an impossibility. It is then that I reviewed carefully in my mind what you had said and got the correct answer.

Mom: Good job.

Yma: But, Mom. Wait a minute. There is something pretty weird going on here.

Mom: How do you mean?

Yma: Well, the light travels further in a given time relative to the ground than it does relative to the train. So, if the light is moving at the same speed relative to both, then we have a paradox: it is covering at the same time, and at the same speed, two different distances.

Mom: Good for you.

Yma: You mean I am right? Then what is the way out of this paradox?

Mom: Well, the answer to that question goes to the heart of what is involved in the study of relativity theory. For example, one of Einstein's famous results is that time runs slower on a moving train than it does from the perspective of the platform. So, the time traveled by the light we have been discussing is different on the train and relative to the platform. This is where you made a mistake in what you said.

Yma: Different? How can that be? I have never noticed any difference in my watch when we have traveled on trains. It doesn't run slow.

Mom: Right you are. But that is because we are not traveling at speeds close to the speed of light. It is in this case that the difference becomes noticeable.

Yma: Hold on now. Is that why on Star Trek, one twin can go away on a space ship traveling close to the speed of light and return to earth without having aged as much as the twin who stayed at home?

Mom: You've got it.

Yma: Wow. I didn't know all of this was behind $E = mc^2$.

Mom: Well, now you know. And there is more, much more. But let's not go any further. A full explanation would require some careful thought experiments, of a kind similar to the ones Einstein performed. Do you know that he reasoned out his theory, that he did not work in a laboratory?

Yma: I have heard that; in fact, he worked in a patent office, didn't he?

Mom: Yes, you are right again.

Let's see if for today we can behave as Einstein did and conduct a thought experiment of our own to derive the result that *moving clocks run slow*.

Two Thought Experiments

Suppose that a train is moving to the left past a stationary observer on a platform.

1. Suppose that on the train, someone turns on a flashlight and directs the beam horizonatlly at the front wall of the coach.

   a. On the platform, the observer measures as t the time it takes the light to travel to the wall. Then explain why the distance the light travels to the wall relative to the platform is ct.

   b. On the train, the length of time it takes the light to reach the wall is measured as t'. Then relative to the train, the light travels a distance ct'. Right?

   c. Note that relative to the platform, the distance the train travels during time t is vt where v is the velocity of the train. Right?

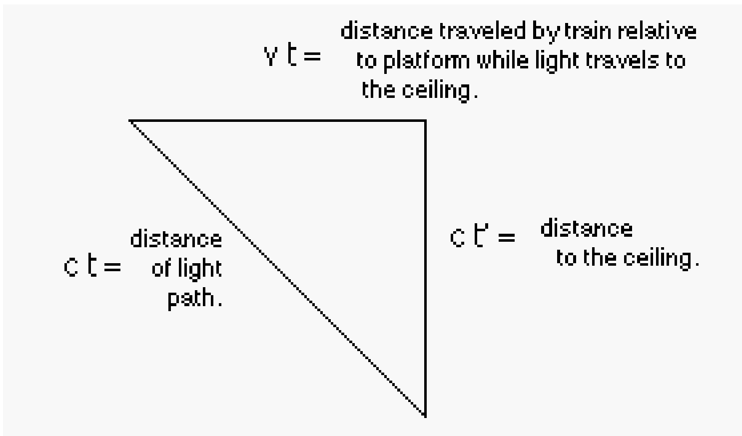
      (i) Explain why the following equation seems reasonable:

      $$ct = ct' + vt$$

      (ii) Next, describe what happens when v is close to c (say, $v = \frac{9}{10}c$). Is the equation still reasonable? Do we have any reason to trust the original equation in (i)? Do we have reason to distrust it?

2. Here is another experiment. Now, the person on the train directs the flashlight vertically at the ceiling. So, the light beam is not shining in the direction of movement of the train.

   a. Similar to before, on the platform the observer measures as t the time it takes the light to travel to the ceiling. The path of the light is shown below. When you have read (below) part b and the beginning of part c, you should be able to explain why the drawing makes sense. So, read on.



b. On the train, the length of time it takes the light to reach the ceiling is measured as t'. Then relative to the train, the light travels a distance ct'. Right?

c. As before, relative to the platform, the distance the train travels during time t is vt where v is the velocity of the train. Right?

   (i) Explain why the distances are related in the following way:

   $$(ct)^2 = (ct')^2 + (vt)^2$$

   (ii) Show that t and t' are related through the equation:

   $$t' = t\sqrt{1-\frac{v^2}{c^2}}$$

   (iii) Explain what happens as $v \to c$.

   (iv) Compare this relationship between t and t' with the relationship you considered in part 1.c. (take $v = \frac{9}{10}c$ for example). Which equation do you trust more and why? Which do you think Einstein proposed?

   (iv) Explain why one conclusion of Einstein's theory of relativity is that *moving clocks run slow relative to a stationary observer*.

Exercise: Suppose identical twins part at age 20. One of them remains on earth while the other spends 10 earth-years traveling in space in a space ship whose speed is $\frac{1}{2}c$. What are the ages of the twins when the ship returns to earth? Answer the same question if the space ship travels at $\frac{9}{10}c$.