Mathematics 5
Winter Term 2008
The World According to Mathematics

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Class Discussion: Week #6

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Today we are going to be discussing pure and applied mathematics—the differences, similarities and interplay between them. We first will look at some mathematical examples, and then we will discuss an excerpt from a book by G.H. Hardy, a twentieth century mathematician.

1. ISBN Error-correcting Algorithm

Here is an algorithm for correcting common errors in transmitted ISBN's.

Theorem: Suppose a transmitted ISBN is recorded as $a - bc - defghi - j$ and its check-sum reports an error $Z \neq 0$:

$$[10a + 9b + 8c + 7d + 6e + 5f + 4g + 3h + 2i + j](\bmod 11) = Z.$$

(a) Off-by-one errors: If one of the digits of the transmitted ISBN is one more or less than it should be, then

one more: the digit with the multiplier Z is wrong (we treat the check-digit j as having multiplier 1).

one less: the digit with the multiplier congruent to $-Z(\bmod 11)$ is wrong.

(b) Transposition errors: If the adjacent transposed digits of the transmitted ISBN are, in order from left to right, u and v, then Z is congruent to $(u - v)(\bmod 11)$.

Answer the questions below.

a. Suppose the actual ISBN is 0-679-79171-X and it is transmitted as 0-679-79181-X. Suppose further that the result of applying the ISBN algorithm to 0-679-79181-X yields Z = 3. Verify that the error-correction theorem above gives the correct information about where the error occurs.

b. Suppose the actual ISBN is again 0-679-79171-X and it is transmitted as 0-579-79171-X. Suppose further that the result of applying the ISBN algorithm to 0-579-79171-X yields Z = 2. Verify that the error-correction theorem above gives the correct information about where the error occurs.

c. Suppose the actual ISBN is again 0-679-79171-X and it is transmitted as 0-679-71971-X. Suppose further that the result of applying the ISBN algorithm to 0-

679-71971-X yields Z = 3. Verify that the error-correction theorem above gives the correct information about where the error occurs.

d. How could you use the above theorem to correct errors in a transmitted ISBN? Explain.

e. What mathematical ideas do you suppose were used to develop the above error-correcting algorithm? (To get you started in your thinking, remember that Gauss introduced the idea of congruence modulo n.) Is there an interplay between pure and applied mathematics? Explain.

2. <u>The Euler φ-function</u>

a. Your groups will be calculating powers of numbers in various mods. Each of you will be given a number, and your group will be given a mod. For example, your group may be given the mod 14, and you may have the number 5. In that case, you would compute powers of 5 modulo 14. Discuss with your group what types of patterns you see.

b. Recall that two integers are *relatively prime* if the only divisor they have in common is ± 1. We then use this notion to define the Euler φ-function (pronounced "fee-function"): If n is a natural number, then $\varphi(n)$ equals the number of integers from 1 to n that are relatively prime to n.

After we discuss the above results in part a. as a class, calculate the following values:

$\varphi(4)$
$\varphi(9)$
$\varphi(25)$

Do you see a pattern?

Test it on:

$\varphi(16)$
$\varphi(36)$
$\varphi(49)$

Discuss your findings with your group. Then calculate:

$\varphi(8) = \varphi(2 \cdot 2 \cdot 2)$
$\varphi(27) = \varphi(3 \cdot 3 \cdot 3)$
$\varphi(125) = \varphi(5 \cdot 5 \cdot 5)$

Discuss these results and see how your group thinks they relate to the earlier findings.

3. <u>A Mathematician's Apology</u>

In 1940, the eminent mathematician G.H. Hardy published A Mathematician's Apology. The book was published subsequently in 1941, 1948, and 1967. We have prepared a handout from this book for today's discussion. The handout is a Xerox of pages 59 and 60, from section 21 of the 1941 edition.

Read this selection and discuss it in light of the above mathematical examples and in light of what you know about pure and applied mathematics. The following quote is particularly pertinent: "The 'real' mathematics of the 'real' mathematicians, the mathematics of Fermat and Euler and Gauss and Abel and Riemann, is almost wholly 'useless'..."