Mathematics 5
Winter Term 2008
The World According to Mathematics

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Friday Discussion: Week #4

Part 1: Geometric Sums

Consider for a positive number r the geometric sum:

$$S_n = 1 + r + r^2 + r^3 + r^4 + r^5 + \cdots + r^n$$

Exercise 1: What is the value of the above sum for $r = \frac{1}{2}$ and $n = 6$?

Exercise 2: By multiplying both sides above by $(1-r)$, verify that:

$$S_n(1-r) = \left(1 + r + r^2 + r^3 + r^4 + r^5 + \cdots + r^n\right)(1-r) = 1 - r^{n+1}$$

[Here are the steps in the multiplication of the right-hand-side by $(1-r)$:

$$\begin{aligned} &\left(1 + r + r^2 + r^3 + r^4 + r^5 + \cdots + r^n\right)(1-r) \\ &\quad = \left(1 + r + r^2 + r^3 + r^4 + r^5 + \cdots + r^n\right) - \left(r + r^2 + r^3 + r^4 + r^5 + \cdots + r^{n+1}\right) \\ &\quad = 1 - r^{n+1} \end{aligned}$$

Is this what you got when you did it?]

Now, dividing through by $1-r$ we get that:

$$S_n = 1 + r + r^2 + r^3 + r^4 + r^5 + \cdots + r^n = \frac{1 - r^{n+1}}{1-r}$$

You should find this formula to be useful when you analyze the formulations of Zeno's paradox that we will study below.

Also, think about what it might mean if, starting with the above formula, we let $n \to \infty$ (that is, we let n get large without bound). In mathematical terms, we describe this process by writing the right-hand side as:

$$\lim_{n \to \infty} \frac{1 - r^{n+1}}{1-r}$$

and by writing the rest as:

$$\lim_{n\to\infty} S_n = 1 + r + r^2 + r^3 + r^4 + r^5 + \cdots + r^n + \cdots$$

<u>Exercise 3</u>: If r is a number such that $0 < r < 1$, what do you suppose is the value of

$$\lim_{n\to\infty} \frac{1 - r^{n+1}}{1 - r} =$$

Hint: Try first calculating $\lim_{n\to\infty} \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} =$

Part 2: Zeno's Paradox

This supposedly is what Zeno said:

> "Achilles cannot overtake a fleeing tortoise because in the interval of time that he takes to get where the tortoise was, it can move away. But even if it should wait for him, Achilles must first reach the halfway mark between them and he cannot do this unless he first reaches the halfway mark to that mark, and so on indefinitely. Against such an infinite conceptual regression, he cannot even make a start, and so motion is impossible."

There actually are two different paradoxes contained in Zeno's statement. We will separate them, calling the first the *Achilles Paradox* and the second the *Dichotomy Paradox*.

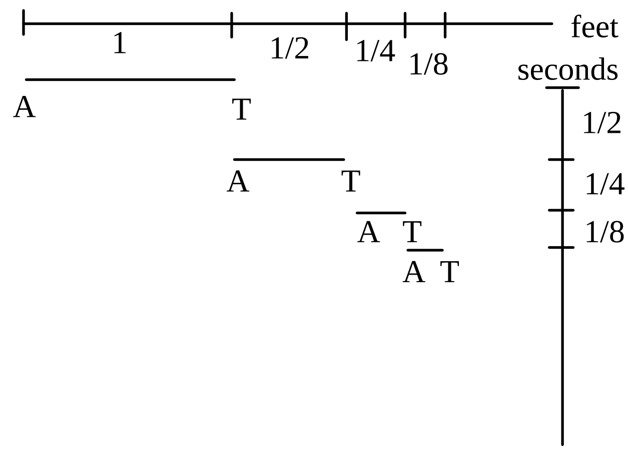
1. Achilles Paradox: Achilles cannot overtake a fleeing tortoise because in the interval of time that he takes to get where the tortoise was, it can move away.

2. Dichotomy Paradox: There is no motion because that which is moved must arrive at the middle before it arrives at the end.

We are going to discuss the Achilles Paradox.

Suppose for sake of simplicity that although they start the race at the same time, the tortoise (T) starts one foot ahead of Achilles (A). Also, assume that the tortoise moves one-half foot, then one-quarter foot, then one-eighth foot, then one-sixteenth foot, and so on. Furthermore, assume that Achilles always moves to the position just vacated by the tortoise. That is, he is behind the tortoise by the amount the tortoise just moved. So, Achilles moves one foot, then one-half foot, then one-quarter foot, then one-eighth foot, then one-sixteenth foot, and so on. This seemingly leads to the paradoxical conclusion that Achilles never overtakes the tortoise.

Let us make one further assumption, relating the movements of Achilles and the tortoise to the passage of time. Assume that the first move of the two occurs in one-half second, the second move in one-quarter second, the third move in one-eighth second, and so on. This seems to lead to the paradoxical conclusion that the race is never ended.



<u>Questions and comments to think about:</u>

1. With reference to the above sketch, do you agree that the following is an expression for the sum of the distances traveled by Achilles? 1 + 1/2 + 1/4 + 1/8 + 1/16 + ….

   What do you suppose the three dots at the end stand for? [Don't worry if you don't know how to do the arithmetic that the statement suggests. We will remedy that below. In this question and the next three, you should just be taking information from the drawing and putting it into rather suggestive mathematical statements.]

2. Write an expression for the sum of the distances traveled by the tortoise:

3. Write an expression for the sum of the intervals of time traveled by Achilles:

4. Write an expression for the sum of the intervals of time traveled by the tortoise:

5. Some mathematical notation can be helpful here: 2^n means multiply 2 times itself n times. So, $2^2 = 2 \times 2 = 4$, $2^3 = 2 \times 2 \times 2 = 8$, $2^4 = 2 \times 2 \times 2 \times 2 = 16$. What is 2^5?

   Also, an arithmetic fact is that $\frac{1}{2^2} = \left(\frac{1}{2}\right)^2$. Check it; both the left-hand and right-hand side of the equation equal 1/4. Similarly, $\frac{1}{2^3} = \left(\frac{1}{2}\right)^3$. Check it. What is the value?

6. Combining what we did earlier with the results in #5, do you agree that we can rewrite our earlier statements as:

   Sum of distances traveled by Achilles: $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$

   $$= 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots$$

   Sum of distances traveled by Tortoise: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$

   $$= \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots$$

7. The above sums (that are unending and go on and on) are both examples of what is called a *geometric series*. They are not at all sums in the usual arithmetic meaning of that word. If the sum stops after a finite number of terms, we call it a *finite geometric sum*. We can add up the terms of a finite geometric sum using the usual rules of arithmetic.

   Replacing 1/2 by *r*, we can develop a general formula for the sum of a finite geometric sum for any r strictly between 0 and 1 as follows: (Do not be concerned if you don't remember your algebra well enough to work out the details establishing the first equation. We will be working with the second formula below. Look at the second formula and see if you recognize what the symbols mean and how to use it. For example, write out the formula when $n = 3$ and $r = 1/2$. You can get a start on this by looking at the bracketed comment following the formula. Notice that the right-hand side of the formula allows you to get the answer while avoiding doing the tedious additions of the fractions on the left-hand side of the equation.)

$$\left(1+r+r^2+r^3+\cdots+r^n\right)\left(1-r\right)=1-r^{n+1}$$

$$1+r+r^2+r^3+\cdots+r^n=\frac{1-r^{n+1}}{1-r}$$

[With $r=1/2$, we have $1+\frac{1}{2}=\frac{1-\left(1/2\right)^2}{1-1/2}$ for $n=1$, and $1+\frac{1}{2}+\left(\frac{1}{2}\right)^2=\frac{1-\left(1/2\right)^3}{1-1/2}$ for $n=2$. Verify these and then go ahead and try writing out the formula for $r=1/2$ and $n=3$.]