Mathematics 5
Winter Term 2008
The World According to Mathematics

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Friday Discussion: Week #3

Part 1: Definitions

1. In mathematics, we start with definitions and use them to state axioms (which are assumed and don't need proof) and theorems (which do require proof). However, a very interesting and important portion of mathematics is choosing the right definition for what we want to talk about. Think about the concept of a *hole* in a surface. We know intuitively what a hole is, but how could we define one rigorously? What would be your definition of a hole?

2. Similarly, sometimes there are many rigorous definitions of the same term, but one that is more helpful for purposes of proving things. Consider the idea of *Greatest Common Divisor* (also called *Greatest Common Factor*). When you learned about this concept, you probably simply said the obvious thing: "The greatest common divisor between two numbers is the largest number that divides evenly into both." However, for many purposes another definition (which is equivalent) works better:

    > Let m and n be natural numbers. Then the greatest common divisor between m and n denoted (m, n) is the unique natural number d such that:
    > i) $d > 0$
    > ii) d divides evenly into both m and n.
    > iii) if c divides evenly into both m and n, then c divides evenly into d.

    As a group verify that this is an equivalent definition, and also discuss why it may be a better one to use for purposes of proof.

Part 2: Conditional Statements and Valid Arguments

1. Here are five statements taken from a book written by Lewis Carroll, author of *Alice in Wonderland*.

    1. No kitten that loves fish is unteachable.
    2. No kitten without a tail will play with a gorilla.
    3. Kittens with whiskers always love fish.
    4. No teachable kitten has green eyes.
    5. No kittens have tails unless they have whiskers.

    (a) Rewrite each of the statements in the *If __, then __.* form.
    (b) Write the statements in (a) in symbolic form p→q.
    (c) Using the Law of Syllogism, [(p→q) and (q→r)] implies (p→r), reorganize the symbolic statements in (b) and deduce the one conclusion that follows from

these statements. [For example, if two of your statements are p→q and q→r, then you can deduce that p→r. Continue in this way with the other statements.]

    (d)   Write your symbolic answer in (c) in words again.

2. Explain why the following method may be used to show that an argument is valid. Determine all cases in which the conclusion is false, and show that in each case at least one premise is false.

Part 3: Linguistic Paradoxes

1. Liar's Paradox:

    Explain why the following sentence is self-contradictory, neither true nor false:

        This statement is false.

2. The Richard Paradox:

    Here is a logical paradox formulated by Jules Richard (a Frenchman) in 1903:

    Suppose we want to make a list of the properties of the counting numbers, that is, of the whole numbers. First, we would have to list the characteristics—characteristics such as *even*, *odd*, *multiple of 7*, or *perfect square*. Then we would have to write out the definitions of the characteristics. For example, we might give the following definitions for the characteristics we have just presented:

        A whole number is *even* if and only if it is divisible by two.

        A whole number is *odd* if and only if it leaves a remainder of one when divided by two.

        A whole number is a *multiple of seven* if and only if it is divisible by seven.

        A whole number is a *perfect square* if and only if it is the product of a whole number with itself.

    Next, we would have to decide a rule for ordering the definitions in the list. That is, how would we decide which to list first, or which to list second, and so on?

    A simple rule for ordering the definitions is based on the number of letters in a definition. Thus, we count the number of letters in all the definitions and assign position number 1 to the definition with the smallest number of letters. We assign position number 2 to the definition with the next smallest number of letters, and proceed in this way to assign positions to all of the definitions. If two or more definitions have the same number of letters, we assign their positions on the basis of the alphabetical order of the letters in each. Therefore, each definition will have its own position number.

    Exercise 1:   Suppose we use the above rule to order the four definitions given earlier. What are their position numbers?

2

Exercise 2: Think of another characteristic of whole numbers and make up a definition for it. Then assign it a position number in the list of five definitions and reorder the other definitions as necessary.

Now, we note the following. For a given position number, either it possesses the characteristic given in the definition to which it corresponds or it does not. For example, suppose 17 has been assigned to the definition of *divisible by five*. Then 17 does *not* have the characteristic described. On the other hand, if 14 has been assigned to the definition *divisible by seven*, then 14 *does* have this characteristic.

Definition: A position number that does *not* possess the characteristic described in the definition to which it corresponds is called *Richardian*.

In the above examples, 17 is Richardian and 14 is not Richardian.

Exercise 3: From Exercise 2, you have assigned five position numbers. Which ones are Richardian and which ones are not?

Exercise 4: Consider the characteristic of being Richardian. Calculate its position number in the list of six definitions—five from Exercise 2 and the definition of Richardian above. We will call this number n so that we can refer to it.

Exercise 5: Now, you should explain why there is a paradox inherent in the following question: Is the number n Richardian? This paradox is called the *Richard paradox*.

Exercise 6: Propose a way out of the Richard paradox based on a theory-of-types kind of argument. That is, is *being Richardian* an arithmetic property or something else?

3. Here is a paradox of Bertrand Russell involving sets. Some sets (taken as a whole) *are not* members of themselves. For example, a set (i.e. collection) of butterflies is not a butterfly. Other sets *are* members of themselves. For example, the collection of all things that are not butterflies is a thing that is not itself a butterfly, and is hence a member of itself. Now, consider the set R, where

R is the set of all sets that are not members of themselves.

Question: Is R a member of itself, or not? Can you propose a way out of this paradox?