Notes for Lecture 1

Definition :A *binary relation,** on a set **S** is a rule that assigns to each ordered pair $(a, b) \in \mathbf{S}$ a value $a * b = v$ such that:

i) $v \in \mathbf{S}$ [We say **S** is *closed* under $*$.]

ii) If $a * b = c$ and $a * b = d$ then $c = d$ [We say this means that $*$ is *well-defined* .]

Example : Addition, multiplication, subtraction, and exponentiation are binary operations on the integers, but division is not.

Definition : A *free semi-group* **F** on a set **S** is the set of all strings of elements in **S** with the binary operation of concatenation.

Notation : We will use the symbol ϕ to denote the "empty string"
.

Example : If $\mathbf{S} = \{\mathrm{a}\}$ then the free semigroup on **S**would have the elements:

$\phi \qquad a \qquad aa \qquad aaa \qquad aaaa, \ldots$

Example : if $\mathbf{S} = \{\mathrm{a}, \mathrm{b}\}$ then all the following would be elements in the free group on **F**:

$\phi \qquad a \qquad b \qquad aa \qquad ab \qquad ba \qquad bb \ldots$

Example : In the above case the following multiplications would occur:

$aa * b = aab$

$a * a = aa$

$\phi * aba = aba$

$abb * ab = abbab$

Definition : A *Finitely-Generated Semigroup* is a set of equivalence classes induced by adding a set of relations to a *Free Semigroup* on a finite set **S**.

Example : Let $\mathbf{S} = \{\mathrm{a}\}$ and let **T** be the *finitely-generated semigroup* on **S** with the set of relations $\mathbf{R} = \{\mathrm{aaa} = \mathrm{a}\}$, then there are exactly 3 elements in $\mathbf{T} : [\phi], [\mathrm{a}], [\mathrm{aa}]$.

Example : Let $\mathbf{S} = \{\mathrm{a}, \mathrm{b}\}$ and let **T** be the *finite-generated semigroup*on **S** with the set of relations $\mathbf{R} = \{\mathrm{aab} = \mathrm{b}, \mathrm{ba} = \mathrm{b}, \mathrm{bb} = \mathrm{b}\}$. Then the elements of **T**are $[\phi], [ab], [b], [a], [aa], [aaa], \ldots$.

Note : The *Free Semigroup* on **S** is the same thing as *Finitely Generated Semigroup* on **S** with set of relations $\{\}$. This is why is called "Free." It is "Free" of relations.

Definition : A *Free Group* **G** on a set **X** is obtained by taking the *free semigroup* on **S**, where **S** has two elements, r and $\hat{r}$, for each $r \in X$, and adding the relations $\{\hat{a}a = a\hat{a} = \phi | a \in X\}$.