SOLUTIONS

Math 5: Music and Sound. Midterm 1

2 hours, 7 questions, 55 points total

Please show your working and pay attention to the indicated number of points available per question. The last page has useful information.

1. [8 points]

   (a) What is the frequency of the pitch E♭6 in the equal-tempered system?

$$\underbrace{A4 \longrightarrow A5}_{\substack{1 \text{ octave} \\ 12 \text{ semis}}} \; \underbrace{A5 \longrightarrow E\flat 6}_{6 \text{ semis}}$$

440Hz.

18 semis total interval.

$$f_{E\flat 6} = 440 \cdot 2^{18/12}$$

$$\approx 1244.5 \text{ Hz}.$$

   (b) What musical pitch is the frequency 350 Hz nearest, and what is the error from this pitch in *cents*?

$$n = 12 \, \frac{\ln \frac{350}{440}}{\ln 2} = -3.962$$

$$\approx -4 \quad \text{so} \quad 4 \text{ semis below A4 is F4.}$$

$$\text{error in cents} = 100(-3.962 - (-4)) = +3.8 \text{ cents sharp.}$$

   (c) Construct the frequency ratio between E and C in the Pythagorean C major (diatonic) scale. (Briefly show working)

fifths are 3:2 ie $3/2$

$$\text{C} \xrightarrow{3/2} \text{G} \xrightarrow{3/2} \text{D} \xrightarrow{3/2} \text{A} \xrightarrow{3/2} \text{E}$$

C D E G A D E

$3/2$ $3/2$ $3/2$ $3/2$

$1/2$

$1 \quad 9/8 \quad 9^2/8^2$

ratio E to C is

$$81/64.$$

2. [7 points + bonus] An ice cream truck produces a constant pure tone at pitch A5 (880 Hz).

(a) Assuming you and the truck are at rest, write a mathematical formula for the pressure signal heard as a function of time t in seconds.

$\omega = 2\pi f = 1760\pi$

$$C \sin(1760\pi t + \phi)$$

C ← unknown ampl. ϕ ← unknown phase.

(b) If you stand still and the truck comes towards you at 41 m/s, what frequency do you hear?

Doppler: Source $\xrightarrow{V}$ • fixed Observer

$$\frac{f_{obs}}{f} = \frac{1}{1 - V/c}$$

$$\text{so } f_{obs} = 880 \frac{1}{1 - \frac{41}{340}} = 1000.7 \text{ Hz}$$

(c) Assume the truck is now stationary. What speed and direction (towards or away from the truck) would you have to be traveling at to lower the pitch to D5? (a perfect fifth; you may use 3:2).

Lowering the pitch means observer moves away from source (v negative)

fixed • Source   Observer $\xrightarrow{V}$

$$\frac{2}{3} = \frac{f_{obs}}{f} = 1 + \frac{v}{c} \quad \text{for moving observer}$$

↑ f_{obs} perfect fifth down from f.

$$\text{so } \frac{v}{c} = -\frac{1}{3} \qquad v = -\frac{1}{3}c = -113 \text{ m/s}$$

ie 113 m/s away from truck.

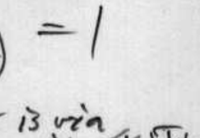
(d) BONUS: You and the truck are stationary but a very strong wind of 34 m/s is blowing in the direction from the truck to you. What frequency do you hear?

880 Hz, since relative to the wind, the source moves at velocity −34 m/s towards the obs, and obs. moves +34 m/s towards source.

wind $\xrightarrow{v}$ • fixed Source • fixed Observer same as (no wind) $\xleftarrow{-V}$ • Source $\xleftarrow{V}$ • Observer

$$\frac{f_{obs}}{f} = \frac{1 + \frac{v}{c}}{1 - \left(\frac{-v}{c}\right)} = 1$$

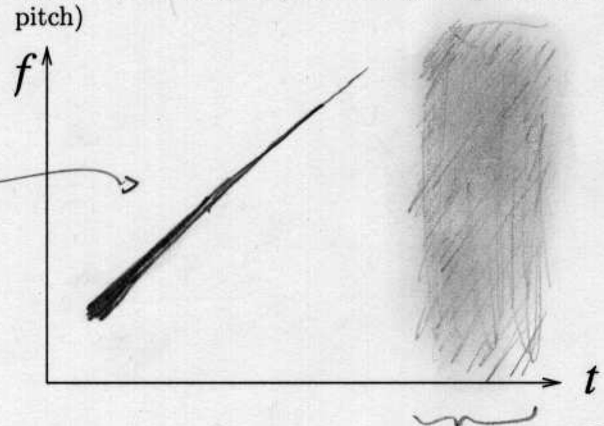
Another way to get it is via spacetime diagram.



3. [7 points] Sketch spectrograms on the axes provided which could realistically match the following sounds. Feel free to explain any features in words too:

   (a) A pure tone rising in pitch but getting quieter, then followed by a hissing sound (no apparent pitch)

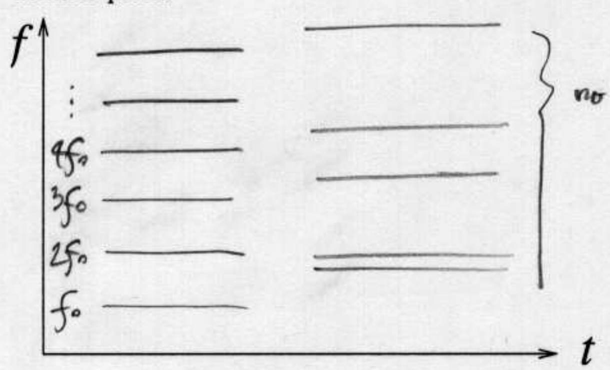
intensity (darkness of line) drops.



hissing has intensity at all freqs. (HW3 you hissed & saw spectrogram).

   (b) A musical note with harsh timbre which is a periodic signal, followed by a bell-like note with no definite pitch.

note strong high harmonics ⇒ harsh.



no obvious integer ratios between partials.

4. [5 points] A bell produces the following partials all at roughly equal amplitudes: 302, 781, 1168, 1560, 2964. What 'strike tone' frequency is perceived, and why?

table of ratios :

	302	781	1168	1560	2964
302		2.59	3.87	5.17	9.8
781			1.496	1.997	3.80
1168				1.336	2.54
1560					1.90
2964					

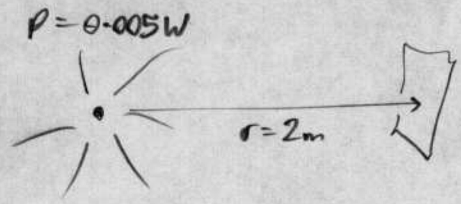
$\approx 3/2$ ≈ 2 $\approx 4/3$

so 781, 1168, 1560 form 2:3:4 ratios (within 1% error, very good)

This is a harmonic series with (missing) fundamental of about 390 Hz (strike tone).

5. [11 points] A tuning fork is struck and produces a pure sinusoid at 300 Hz. A listener is a distance 2 meters from the tuning fork.

(a) Initially the tuning fork radiates 0.005 W acoustic power in all directions. What intensity in dB does the listener hear?



$$I = \frac{P}{4\pi r^2} = \frac{0.005}{4\pi 2^2} = 9.94 \times 10^{-5} \ \mathrm{W/m^2}$$

(or 1.0×10^{-4} to 1% accuracy).

$$dB = 10 \log_{10} \frac{I}{I_r} = 10 \log_{10} \frac{1.0\times10^{-4}}{10^{-12}} = 10(8) = \underline{80 dB}.$$

(b) The Q-factor of the tuning fork is 1000. What is the decay time?

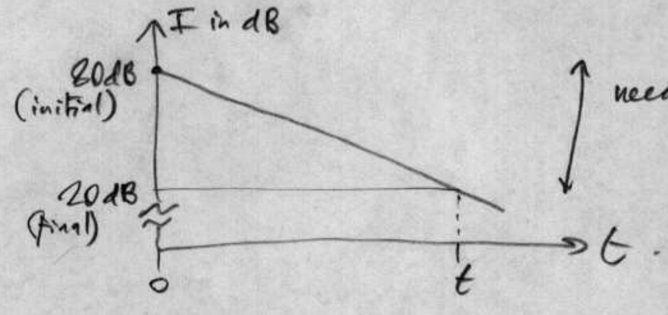
$$Q = \pi \frac{\tau}{T} \qquad T = \frac{1}{f} = \frac{1}{300}$$

rearrange

$$\tau = \frac{QT}{\pi} = \frac{1000 \cdot \frac{1}{300}}{\pi} = \underline{1.06 \text{ sec}}$$

(c) How long since it was struck with the above initial strength does it take until the intensity at the listener reaches the lower threshold of human hearing which is about 10^{-10} W/m^2 at 300 Hz? (careful, not 10^{-12} W/m^2)

is 20 dB since factor 100 above $I_r = 10^{-12}$ W/m^2.



need 60 dB drop.

ie factor 10^6 in intensity,

or 10^3 in amplitude

$$\rightarrow 10^{-3} = e^{-t/\tau} \quad \text{take ln}$$

$$\ln(10^{-3}) = -t/\tau \Rightarrow t = -\tau \ln(10^{-3})$$

$$= \underline{7.3 s}$$

(d) If mass is added to the prongs of the tuning fork so that their *effective* mass doubles (viewing the fork as a mass-spring oscillator), what frequency does the fork sound now?

natural freq. $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ for mass-spring oscillator

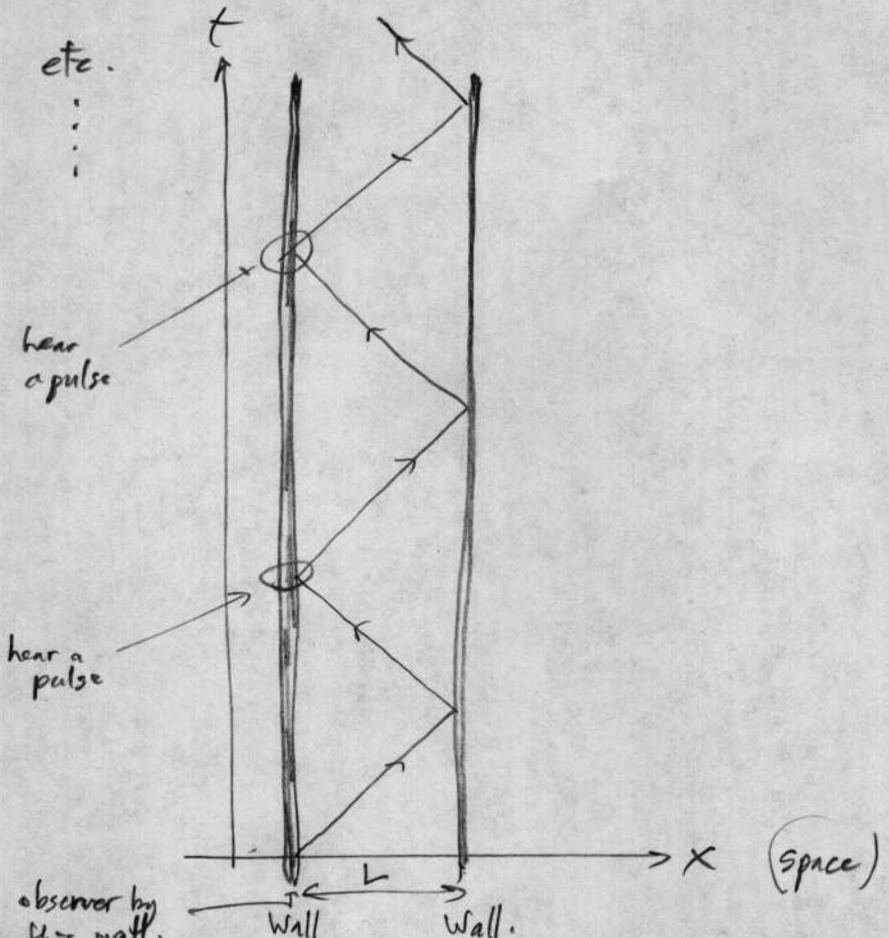
$m_2 = 2m_1$ is doubling the mass.

ratios : $\frac{f_1}{f_2} = \frac{\frac{1}{2\pi}\sqrt{\frac{k}{m_1}}}{\frac{1}{2\pi}\sqrt{\frac{k}{m_2}}} = \sqrt{\frac{m_2}{m_1}} = \sqrt{\frac{2\cancel{m_1}}{\cancel{m_1}}} = \sqrt{2}$

so $f_2 = \frac{1}{\sqrt{2}} f_1 = \frac{1}{\sqrt{2}} 300\text{Hz} = \underline{212.1 \text{ Hz}}$

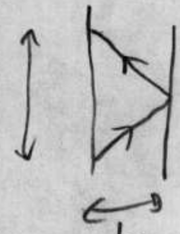
6. [7 points + bonus]

(a) Draw a space-time diagram showing why a flutter echo is heard by a listener standing beside a wall a distance L from another wall, when they produce a short sound such as a clap. Label the walls and any sound pulses.



(b) What period of signal is heard if the spacing between the walls is $L = 10$ meters?

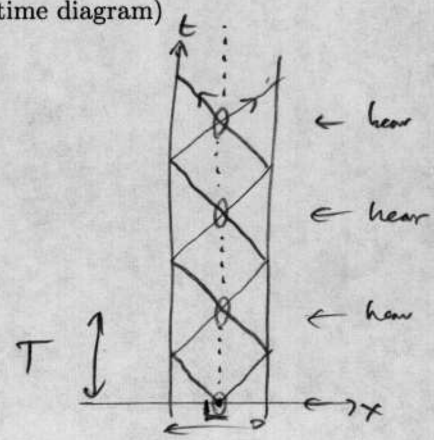
period T is return time of one bounce



pulse travels distance $2L$ speed $c = 340$ m/s.

$$T = \frac{\text{dist}}{\text{speed}} = \frac{2L}{c} = \frac{20}{340} = \underline{0.059\text{ s}}.$$

(c) BONUS: If the listener instead stands *half way* between the walls, what periodicity is heard? (draw a space-time diagram)



period is half what it was before

$$T = \frac{L}{c} = 0.029\text{ s}.$$

7. [10 points + bonus] Random short-answer questions.

(a) Explain briefly what is a Fourier series (you may use an equation). What kinds of functions does it apply to?

ie, signals.

If $g(t)$ is any periodic signal, period T, we may write

$$g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\frac{2\pi n t}{T}$$

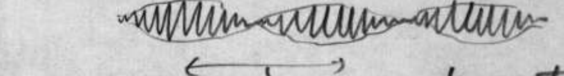
or $n\omega t$, where $\omega = 2\pi f = \frac{2\pi}{T}$.

A periodic function can be written as a sum of sinusoids.

(b) State as clearly as you can what you will hear when pure tones of frequencies 500 and 507 Hz are played together, giving any relevant new frequencies.

$$\left.\begin{array}{l} f_1 = 500 \\ f_2 = 507 \end{array}\right\} \quad |f_1 - f_2| = 7\text{Hz} < 15\text{Hz} \Rightarrow$$ you hear the beat freq. as an amplitude variation.

tone which is modulated is at $\frac{f_1+f_2}{2} = 503\frac{1}{2}$ Hz.

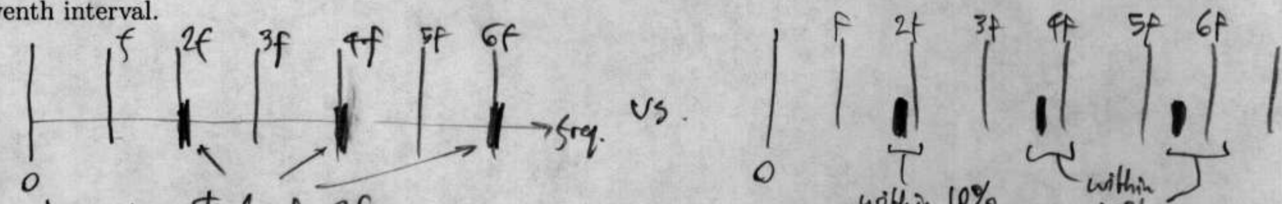


wah repetition at 7Hz.

(c) What frequency has a wavelength about equal to the size of the human ear (4 cm)?

$$c = f\lambda \quad \text{so} \quad f = \frac{c}{\lambda} = \frac{340}{0.04} = 8500\text{Hz}$$

(d) According to the Helmholtz theory, state briefly why an octave is less dissonant than the major seventh interval.



harmonic partials of 2f are 2f, 4f, 6f, fall on top of original partials ⇒ consonant.

Lots of dissonant pairs of partials.

(e) BONUS: When sounds are heard together their signals add. We learned that doubling the amplitude of a signal caused intensity to be multiplied by 4. However when two violins (for example) play the same music together the intensity is merely doubled relative to one violin. Resolve the paradox!

meaning, precise waveforms & phases line up.

'coherent addition' → If exactly the same signals are added, amplitude doubles $A \to 2A$ and intensity quadruples $I \to 4I$.

'incoherent addition' (usual case) → But real musical instruments do not line up crests & troughs of signals exactly (frequencies differ slightly) so amplitude does not double. Consider $\sin(\omega t) + \sin(\omega t + \frac{\pi}{2})$ which has combined amplitude of $\sqrt{2}$ not 2. Averaging over all phase differences ϕ gives $I \to 2I$.