Barnett
6/2/07.

~ SOLUTIONS ~

Math 5: Music and Sound: Final

3 hours, 10 questions, 80 points total

Try to show working. Heed the points available for each question. Try the bonuses once the rest is ok. The last page has useful information. Good luck, have fun, and it was great to have you in the course!

1. [8 points] Tuning systems.

   (a) 1000 Hz is often used as a reference in acoustics. What equal-tempered musical pitch is this nearest? Also compute its error in cents from this pitch.

$$n = 12\,\frac{\ln \frac{1000}{440}}{\ln 2} = 14.21 = \overbrace{12}^{1 \text{ octave}} + 2.21$$

21 cents sharp of B5

   (b) In the C major (diatonic) Pythagorean scale, construct the frequency ratio between C and the A above it.

1/2

C D E F G A B C D

3/2 3/2 3/2

$1 \quad 9/8 \quad \boxed{27/16}$

$\boxed{27 : 16}$

   (c) If you continued using this Pythagorean construction to compute *all* notes of the chromatic scale, what error in cents would occur when you eventually returned to (and compared against) your starting note?

Need to go up perfect fifth 12 times ie $(3/2)^{12} = 129.75$

then go down 7 octaves ie $2^7 = 128$

So ratio between C and the approximation to C obtained by going around the cycle of fifths is $\frac{(3/2)^{12}}{2^7} = \frac{129.75}{128} = 1.0136 \rightarrow c = 1200\,\frac{\ln(1.0136)}{\ln 2}$

$= 23.5$ cents sharp

2. [8 points] Sound pollution. When heard from 90 m away a jet engine produces a sound intensity of 0.1 W/m^2.

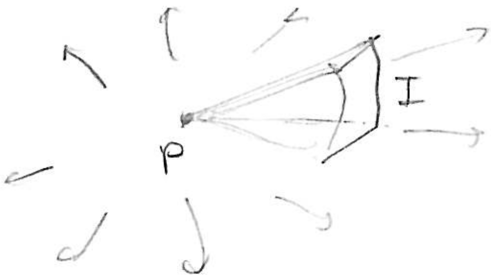
(a) How many dB would be measured at this distance?

$I = 0.1\ W/m^2$

$dB = 10 \log_{10} \frac{I}{I_r} = 10 \log_{10} \frac{0.1}{10^{-12}} = 10 \log_{10} 10^{11}$

$= 110$ dB.

(b) Assuming that sound is emitted uniformly in all directions, how much total sound power is the engine radiating?



$P = 4\pi r^2 \cdot I$ $\quad r = 90m$

$= 4\pi (90)^2 \cdot (0.1)$

$= 10179$ W or 10.2 kW

A lot!

(c) What is its loudness in dB heard from the ground while the plane is flying overhead at a typical cruising altitude of 10 km? (Assume the atmosphere is uniform, and ignore the motion of the plane, reflection from the ground, etc.)

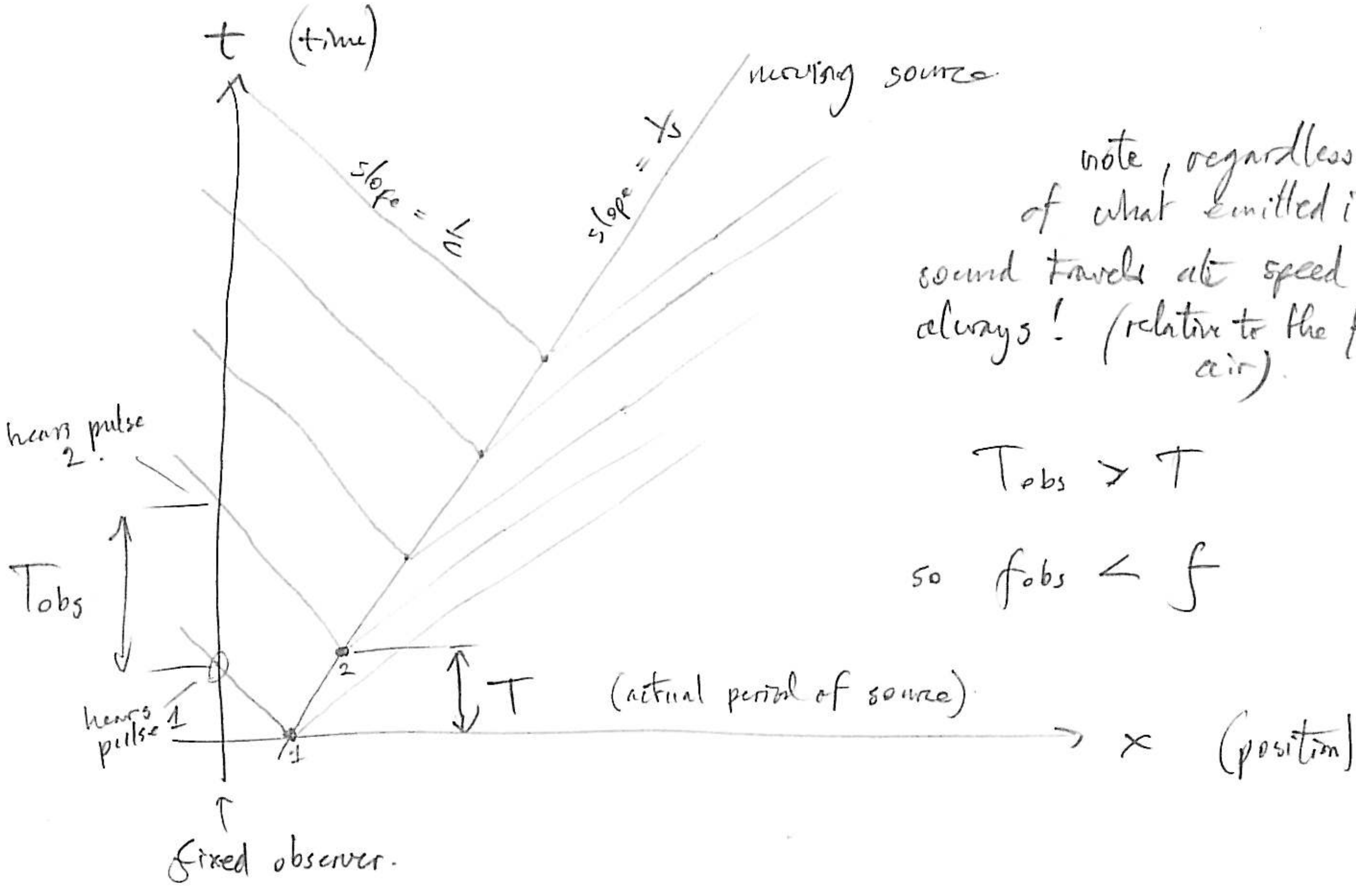
$I = \frac{P}{4\pi r^2} = \frac{P}{4\pi 10^8} \approx 8.1 \times 10^{-6}\ W/m^2$

$r = 10^4 m = 10 km$.

so $dB = 10 \log_{10} \frac{8.1 \times 10^{-6}}{10^{-12}} = 69.1$ dB

3. [7 points + bonus]

(a) Draw a space-time diagram (labeling your two axes) demonstrating why and how the apparent frequency of a periodic sound changes when the source is moving *away* from a fixed observer. Be sure to indicate precisely which quantities in your diagram should be compared to reach your conclusion.



note, regardless of what emitted it, sound travels at speed c always! (relative to the fixed air).

$T_{obs} > T$

so $f_{obs} < f$

(b) What change in pitch (relative to the pitch with source fixed) is observed when the source moves away at the speed of sound? Express as a musical interval.

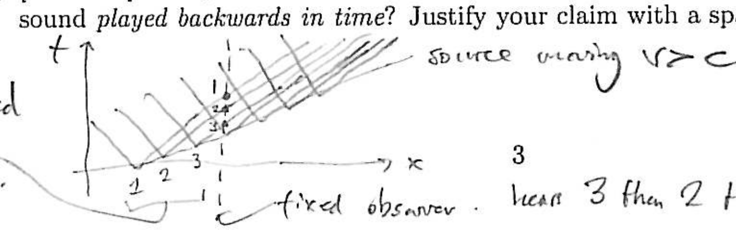
$$\frac{f_{obs}}{f} = \frac{1}{1 - \frac{v}{c}} \quad \text{with} \quad v = -c \quad (\text{speed of sound, away})$$

$$= \frac{1}{1-(-1)} = \frac{1}{2} \quad \text{so 1 octave down.}$$

(c) [BONUS:] Is it possible for a source to move in such a way that the observer hears the emitted sound *played backwards in time*? Justify your claim with a space-time diagram.

Yes (I think) if source moves faster than sound towards you. Eg $v = 2c \Rightarrow f_{obs} = -f$

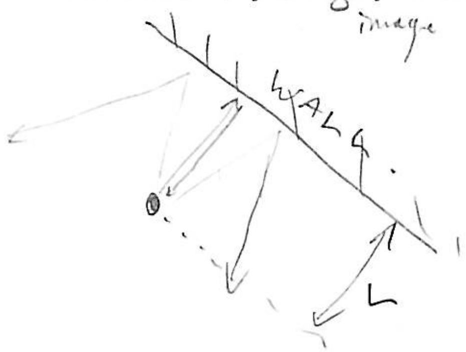
sound emitted in order 1 then 2 then 3.



hear 3 then 2 then 1

4. [7 points] Echoes.

(a) You stand somewhere with your eyes closed, clap, and hear one echo, coming from a definite direction, which is measured to have a delay of 1/10 s after the emitted clap. What type of structure is likely to be in your vicinity and how far away is it?



a single flat wall.

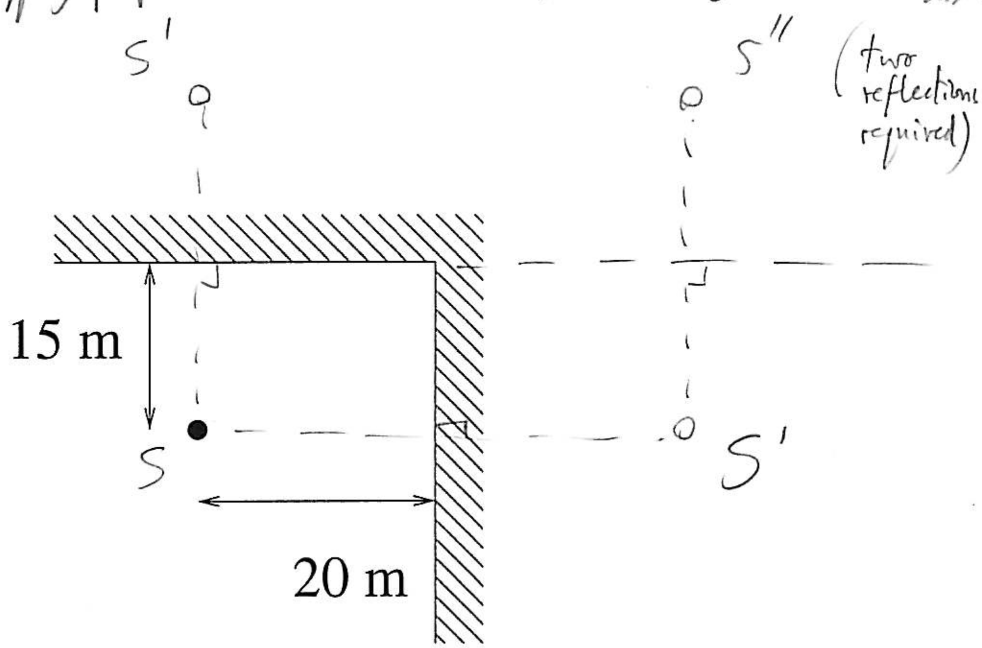
$$\frac{\text{dist trav.}}{\text{speed}} = \frac{2L}{c} = 0.1\,s.$$

$$\text{so } L = \frac{0.1\,c}{2} = \underline{17\,m} \text{ away.}$$

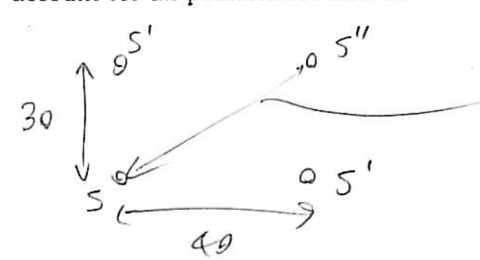
(b) You now stand near two flat walls at right-angles, at the location of the dot shown below, clap and listen as before. Show all *image* source(s) on the diagram.

S' are reached by dropping perpendicular to each wall, continuing behind same dist.

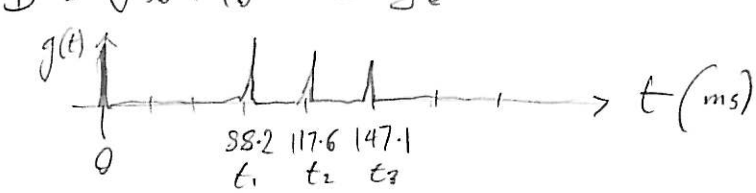
S'' is reached by reflecting either of S' through the (extension of) the other wall.



(c) Use the image source method to predict the delay time(s) of all echo(es) you hear. Be sure to account for all possible reflections.



$$D = \sqrt{30^2 + 40^2} = 50$$



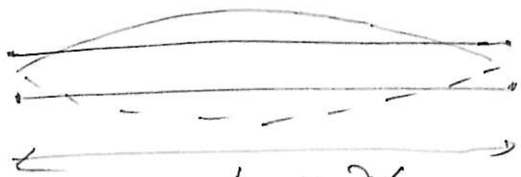
$$t_1 = \frac{30\,m}{340\,m/s} = 0.0882\,s$$

$$t_2 = \frac{40}{340} = 0.1176\,s$$

$$t_3 = \frac{50}{340} = 0.1471\,s$$

5. [8 points + bonus] Wind instruments.

(a) You want to design an organ (behaving as an open-open pipe) producing the note C2 in the equal-tempered system. How long should the pipe be?



$L = \lambda/2$

f_{C2} = 2 octaves + 9 semis below A440

$= 2^{\frac{(-24-9)}{12}} \cdot 440$

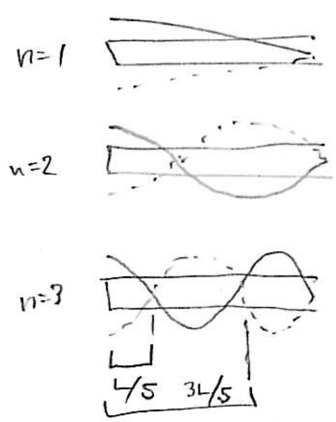
$= 65.4$ Hz.

$L = \frac{\lambda}{2} = \frac{c}{f} \cdot \frac{1}{2} = 2.60$ m.

(b) As you know, the clarinet behaves like an open-closed pipe, and the usual 'registers' involve playing the $n = 1$ and $n = 2$ modes. However for the highest notes the player takes the pipe into its $n = 3$ mode. Compute the frequency ratio and musical interval that occurs when jumping from $n = 2$ to $n = 3$ (without changing the fingering or effective length).

$f_n = (2n-1)\frac{c}{4L}$ with $n = 1, 2, 3, \ldots$

pressure graphs:



ratio $= \frac{f_3}{f_2} = \frac{(6-1)\,c/4L}{(4-1)\,c/4L} = 5/3$ whatever L is.

$= 8.84$ semitones $= s$

using $s = 12 \frac{\ln 5/3}{\ln 2}$

The interval is major 6th. (16 cents flat of this)

(c) [BONUS:] Find possible location(s) of a register key which when opened would help select the $n = 3$ mode (actually this is done by 'cross-fingering')

open register key at $L/5$ or $3L/5$ from closed end (mouthpiece)

6. [7 points + bonus] Physical properties of sound.

(a) An outdoor public-address system is wanted which sends as narrow as possible beam of sound towards a distant audience. The designer chose a single speaker with a horn which opens to a width of 0.2 m. What frequency range is likely to be audible in a wide variety of directions?

freqs less than that with $\lambda = 0.2m$ ie 1700Hz spread out. $\lambda < D$

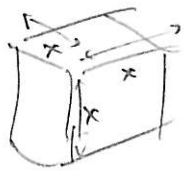
At the highest audible frequencies of 20000 Hz, what limitations on the beam width will there be? (Explain what units you give your answer in)

Beam can be no narrower than 'diffraction limit'

which is $\Delta\theta = \frac{\lambda}{D} = \frac{c}{f}\frac{1}{D} = 0.085$ radians

($\simeq 4.9$ degrees).

(b) What dimensions should a cubical room with absorbtion coefficient 0.05 have in order to achieve the (cathedral-like) reverberation time of 5 seconds?



$V = x^3$

$A = 6x^2$ (six faces each area x^2)

$$T_{60} = 0.161 \frac{V}{S} = 0.161 \frac{V}{\alpha A}$$ for single material type.

$$= 0.161 \frac{x^3}{\alpha \cdot 6x^2} = \frac{0.161}{6\alpha} \cdot x = (0.537) x$$

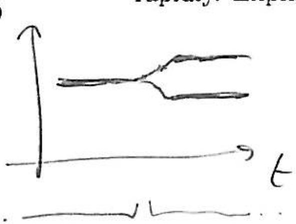
cancel x^2.

want 5sec

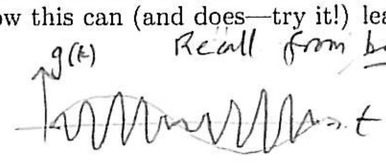
so $x = \frac{T_{60}}{0.537..} = 9.3$ m (≈ 30 feet on a side)

Note: NOT the Doppler effect since doesn't depend on how far apart tuning fork tines are. (Doppler speed v would).

(c) [BONUS:] When struck, a hand-held tuning fork radiates sound strongly in some directions and not at all in others. So when you spin it on its long axis this causes the amplitude to oscillate rapidly. Explain how this can (and does—try it!) lead to *two* distinct perceived pure tones.



Recall from beats:



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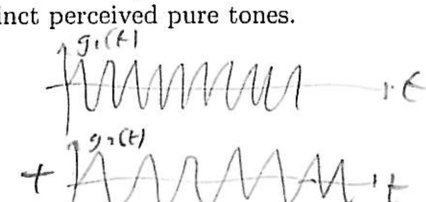


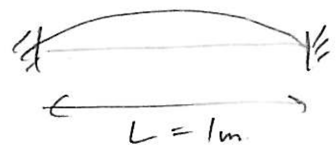
sinusoidal modulation of amplitude

This mathematically is given by addition formula.

two slightly different pure tone sinusoids.

7. [10 points] Assume a piano string has a mass per unit length of 0.001 kg/m, and is 1 m long.

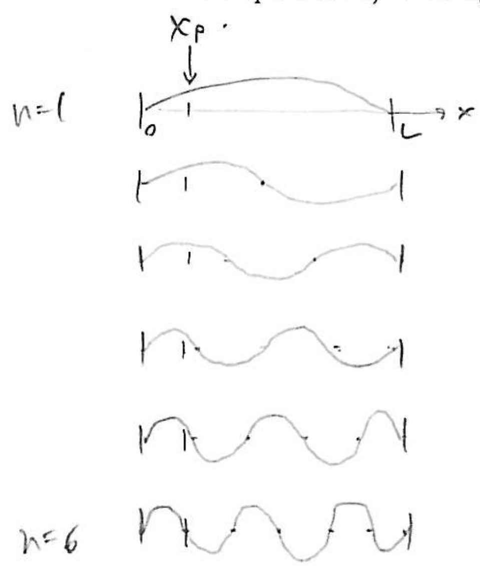
(a) If the tension is 1000 N, find the fundamental frequency of this string.



$$c_{string} = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{1000}{0.001}} = 1000\,\text{m/s}.$$

$$f_1 = \frac{c_{string}}{2L} = 500\text{ Hz}$$

(b) If the string is struck by the hammer 1/6 of the way along, find the excitation amplitudes α_1 to α_6 of the first six modes of the string (preferably give exact numbers; failing this use diagrams to compare them). Are any of them zero? Sketch the resulting frequency spectrum heard.



using $\boxed{\sin \frac{j\pi x_p}{L} = \alpha_j}$

it's fine if you included the $\frac{1}{j}$ factor too!

$\alpha_1 = 1/2$

$\alpha_2 = \sqrt{3}/2$

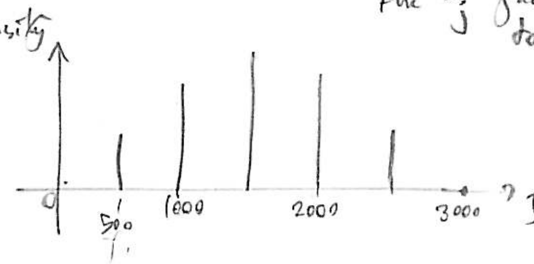
$\alpha_3 = 1$

$\alpha_4 = \sqrt{3}/2$

$\alpha_5 = 1/2$

$\alpha_6 = 0$

only one which is zero.

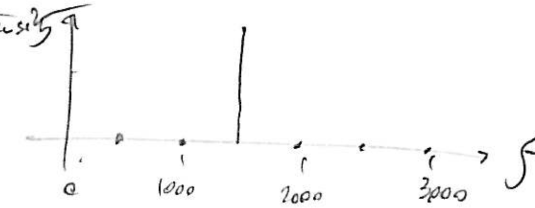


(c) As a special effect in modern classical or jazz (*e.g.* Chick Corea) the pianist reaches inside and lightly touches the string while playing. If they touch the string 1/3 of the way along, while the hammer hits as before, give the new set of α_1 to α_6. Which are zero? Sketch and explain the new frequency spectrum.

touching at $L/3$ kills all modes without a Node there, ie all but n = multiple of 3.

so $\alpha_1 = \alpha_2 = \alpha_4 = \alpha_5 = \alpha_6 = 0$

$\alpha_3 = 1$ (as before).



(d) For many notes, pianos have a *pair* of strings tuned close in pitch. If a piano tuner hears a beat frequency of 1 Hz between two strings that are supposedly both tuned close to 440 Hz, compute the *percentage change* in tension required to bring down the higher pitch string into unison with the lower one.

beat freq $= |f_2 - f_1|$ so strings are 1 Hz apart.

This is $\frac{1}{440}$ fractional error, ie ratio of $\frac{441}{440}$.

Recall $f \propto c_{string} \propto \sqrt{T}$ (μ = const for both)

so $\frac{T_2}{T_1} = \left(\frac{f_2}{f_1}\right)^2 = \left(\frac{441}{440}\right)^2 = 1.0046$ ie -0.46% change to lower the higher string.

8. [11 points] Traditional "udu" drums from Nigeria are large earthenware containers with a hole in the top (in fact they have another hole on the side but let's ignore that). They act as a Helmholtz resonator. Assume the resonant frequency is 50 Hz.

(a) If the neck has area 5 cm^2 and (effective) length 5 cm, what must be the volume of the chamber? (it is fine to quote in m^3)

$$f_{Helm} = \frac{c}{2\pi}\sqrt{\frac{a}{V\ell}}$$

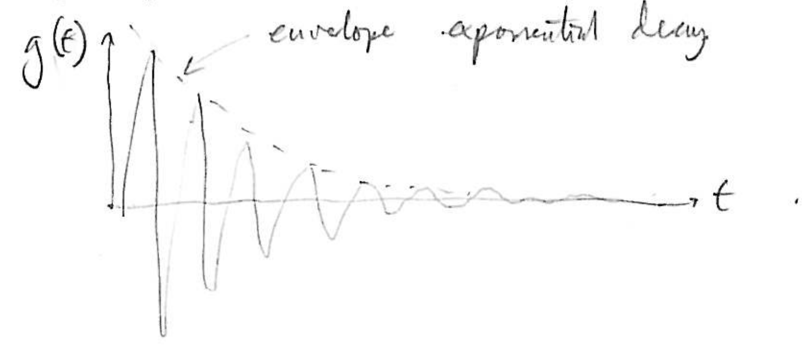
so $$\frac{a}{V\ell} = \left(\frac{2\pi f_{Helm}}{c}\right)^2$$

11.7 liters.

$$V = \frac{a}{\ell}\left(\frac{c}{2\pi f_{Helm}}\right)^2$$

$$= \frac{0.0005\ m^2}{0.05\ m}\left(\frac{340}{100\pi}\right)^2 = 0.0117\ m^3$$

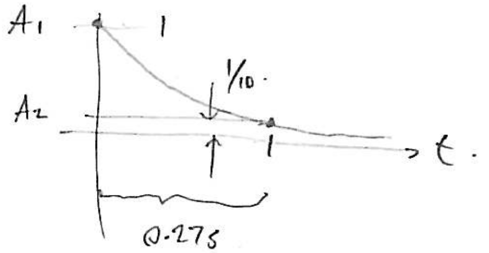
(b) When impulsively excited (by a single 'slap') the drum produces a pure tone which decays exponentially in amplitude. Sketch a graph of the sound signal you expect to record.



(c) If it takes 0.27 sec for the loudness to drop by 20 dB, compute the decay time.

so $-20\,dB = 10 \log_{10} \frac{I_2}{I_1}$ so $\frac{I_2}{I_1} = 10^{-2}$

$\frac{A_2}{A_1} = \sqrt{\frac{I_2}{I_1}} = 0.1$ ← exponential decay causes this change



$\Rightarrow e^{-t/\tau} = \frac{1}{10}$ with $t = 0.27s$.

take logs.

$-\frac{0.27s}{\tau} = \ln \frac{1}{10}$ so $\tau = \frac{0.27}{-\ln 1/10}$

$= 0.117s$

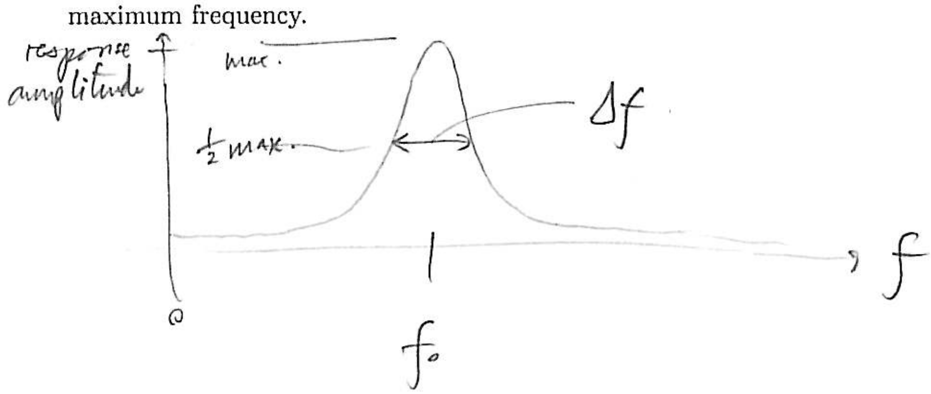
(d) What is the Q-factor of this mode?

$Q = \pi \frac{\tau}{T}$ ← decay time, ← period.

$T = \frac{1}{f_0} = 0.02s$. ($f_0 \approx 50Hz$)

$= 18.4$.

(e) When you place your ear near the opening, ambient frequencies near 50 Hz will sound amplified (boosted). Compute the *range* of frequencies that will be boosted at least half as much as the maximum frequency.



$Q = \frac{f_0}{\Delta f}$

so $\Delta f = \frac{f_0}{Q} = 2.7$ Hz.

so range is $[f_0 - \frac{\Delta f}{2}, f_0 + \frac{\Delta f}{2}]$

$= 48.6$ Hz to 51.4 Hz.

[About 1 semitone in width].

9. [7 points] Signals and such.

$\omega = 2\pi f$

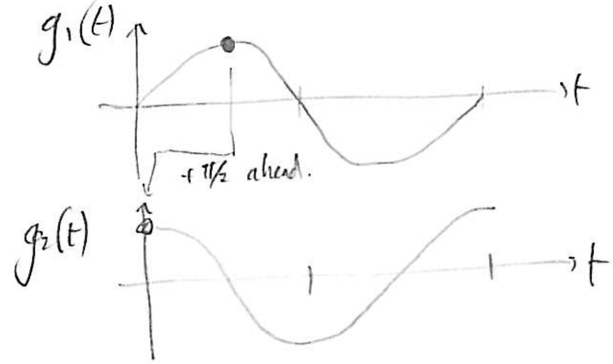
(a) What is the period of the signal $\sin(\boxed{2000}t + \pi/2)$?

$\sin(\omega t + \phi)$ has $T = \frac{1}{f} = \frac{2\pi}{\omega}$

$= \frac{2\pi}{2000} = \frac{\pi}{10^3}$

$= 0.00314$ s

(b) Two pure tones of exactly the same frequency are heard simultaneously, both with amplitude 1, but the second one ahead in phase by $\pi/2$ (90°) relative to the first. What amplitude and phase (relative to the first tone) will the combined signal have? [Hint: Recall the relative phases of your standard trig functions].



ampl.

$1 \cdot \sin(\omega t)$

$1 \cdot \sin(\omega t + \pi/2) = 1 \cdot \cos(\omega t)$

$A \sin\omega t + B\cos\omega t = C \sin(\omega t + \phi)$ with $C = \sqrt{A^2 + B^2}$

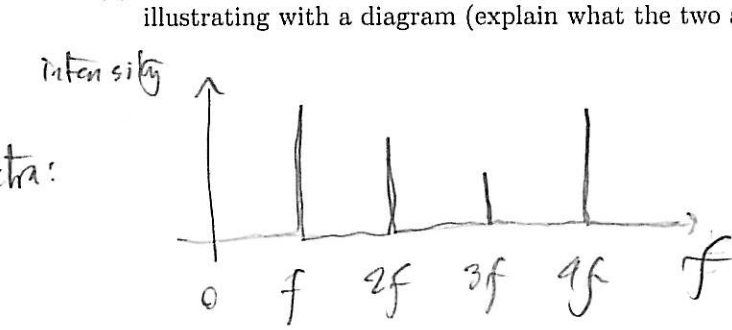
$= \sqrt{1^2 + 1^2}$

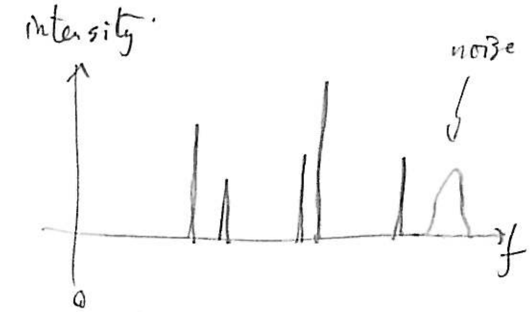
$= \sqrt{2}$

and $\phi = \tan^{-1}\left(\frac{B}{A}\right) = \tan^{-1}(1) = \pi/4$ (45°)

(c) Describe how the frequency spectrum of periodic signals differs from that of non-periodic signals. illustrating with a diagram (explain what the two axes are).

spectra:





periodic

(by Fourier series which involves integer multiples of fundamental only).

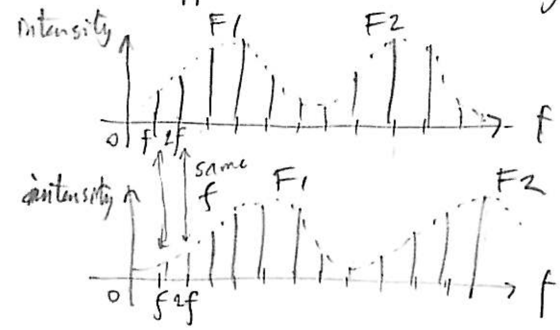
non - periodic.

(may have any spectrum with any peak locations, or broad bands of power).

10. [7 points + bonus] Explain the following paradoxes. You may use words, equations, diagrams.

(a) If a man and a woman speak (or sing) at exactly the same pitch, you can still tell the difference. Why? Show how the sounds (spectra) probably will differ.

They have same set of component partials (since must all be multiples of pitch f_0 = fundamental), but strengths of partials differ due to formant structure:

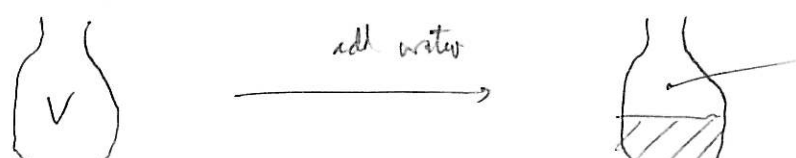


f = pitch.

(F1, F2 about 15–20% higher on average).

(b) As demonstrated in class, the resonant frequency of a bottle goes *up* when you add water, but the resonant frequency of a flat-bottomed metal bowl goes *down* when you add water. Why?

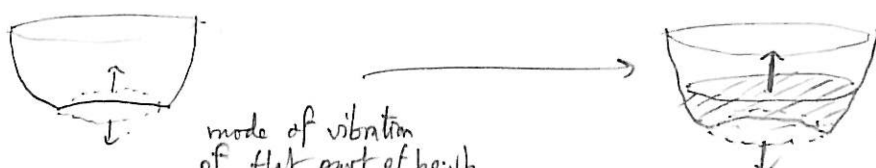
Helmholtz:



volume now smaller.

$\Rightarrow$ freq. $= \frac{c}{2\pi}\sqrt{\frac{a}{Vl}}$ increases

Metal bowl: (is mass-spring oscillator, effectively)

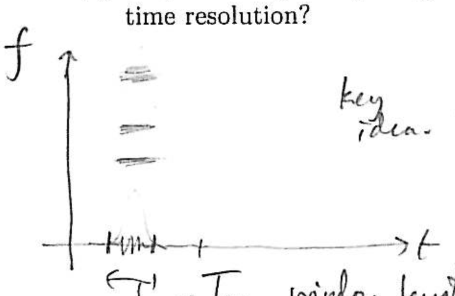


mode of vibration of flat part of bowl.

effective mass m has increased

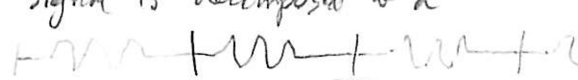
$\Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ decreases.

(c) Why does a spectrogram give worse frequency resolution if you change the settings to have better time resolution?



T_w window length.

key idea. → in each window, signal is decomposed to a Fourier series.



(by treating as one period of periodic signal)

Spacing of Fourier freqs. is $f_0 = \frac{1}{T_w}$ so increases if T_w reduced.

(d) [BONUS:] Roughly what is the equation relating time resolution Δt and frequency resolution Δf in any spectrogram?

$$\Delta t \cdot \Delta f \approx 1$$

"Heisenberg uncertainty relation".

$\Delta t \approx T_w$ window length.

Δf roughly $f_0 = \frac{1}{T_w}$.