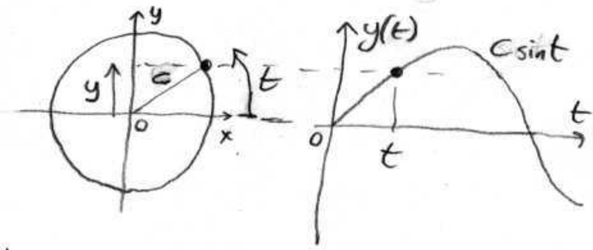
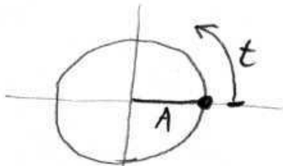
① 3/30/07
Barnett

Everything about additional formulae & adding sinusoids (of the same frequency).

Key [if a point rotates with angle t on circle radius C, its y-coord is $C \sin t$. This is a sinusoid function of t.

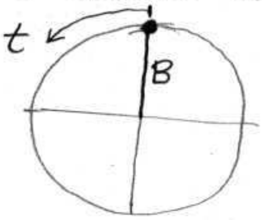


Consider a rotating vector length A which starts horizontal at $t=0$:



its y-coord is $A \sin t$

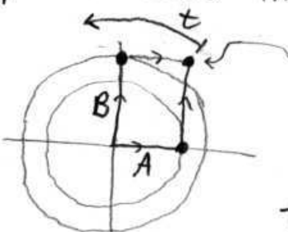
Also consider another rotating vector length B which starts vertical at $t=0$:



its y-coord is $B \cos t$

since it's vertical at $t=0$, it's $\pi/2$ ahead of sin

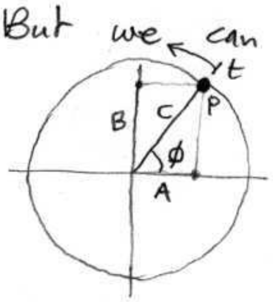
If we add these vectors they remain at right-angles since they rotate at same rate:



this point P is the vector sum (place vectors head to tail)

The whole rectangle rigidly rotates, as if drawn on a turntable

But we can also see point P moves on a larger circle, let's call radius C, and that it is ahead by a fixed angle, let's call it ϕ.



P starts at angle ϕ and rotates, so its angle is $t+\phi$

so P's y-coord is $C \sin(t+\phi)$

This y-coord must equal the sum of the original two y-coords:

$$\boxed{C \sin(t+\phi) = A \sin t + B \cos t}$$

But A, B given by trigonometry! $A = C \cos\phi$, $B = C \sin\phi$ since

right triangle.

So now we know how to get constants A, B which tell us how to break down a phase-shifted sinusoid $\sin(t+\phi)$ into <u>weighted</u> sum of $\sin t$ and $\cos t$.

How do you go backwards? Want C, ϕ given A, B

use the right triangle: $C = \sqrt{A^2+B^2}$

$\phi = \tan^{-1}(B/A)$

Ex. write $\sin(\omega t) + 2\cos(\omega t)$ in the form $C\sin(\omega t + \phi)$?

We have $A=1$, $B=2$ so $C = \sqrt{1^2+2^2} = \sqrt{5}$, $\phi = \tan^{-1}(2/1) \approx 1.11$

Notice this worked for ωt as well as t.

If we take the boxed formula above, substitute in for A, B, the C's cancel:

$$\boxed{\sin(t+\phi) = \cos\phi \sin t + \sin\phi \cos t}$$

This 'addition formula' for sin applies to <u>any</u> numbers t, ϕ.

Note there are also addition formulae for cos, exp, etc.

eg $e^{t+\phi} = e^t e^\phi$ nice and simple

...but not for logarithm: $\ln(t+\phi) = ?$ nothing useful.

If we change ϕ to $-\phi$ in addition formula, and use $\cos(-\phi) = \cos\phi$, $\sin(-\phi) = -\sin\phi$

get $\sin(t-\phi) = \cos\phi \sin t - \sin\phi \cos t$

Add this to the original to get

$$\sin(t+\phi) + \sin(t-\phi) = 2\cos\phi \sin t$$

Substitute $a = t+\phi$, $b = t-\phi$ to get

$$\boxed{\sin a + \sin b = 2\cos\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)}$$

(there are many similar other formulae).

TAKE-HOME MESSAGE: any sinusoids of the same frequency can be added as if they were 2D vectors in the plane. Their amplitude is length, their phase is angle.