Math 56 Compu & Expt Math, Spring 2014: Topics Weeks 1-3 (Midterm 1)

1 Week 1

Relative vs absolute error

Big O, little o. Know definitions, be able to test if one func is O or o of another, as some parameter goes large or small.

Algebraic convergence, order. Know how to bound tail of algebraic series by integral.

Exponential convergence, rate. How to bound tail by pulling out a geometric series. Thm that rate is asymptotically (dist from eval pt to center)/(nearest dist of singularity to center)

How to choose good axes for a plot so data spread and linear, interpret slope.

Definition of superexponential convergence.

Basic complex arithmetic, magnitude-phase notation.

2 Week 2

Taylor's theorem with correct remainder term, using it to bound err. eg using to prove super-exp conv for exp(x)

Newton's iteration. Definition of quadratic convergence (ie $\varepsilon_{n+1}/\varepsilon_n^2 \le C$), sketch of proof that Newton's is quadr conv. Newton's for computing sqrt. Bisection alg from HW.

Set of floating point numbers, their gaps, error due to rounding ie $f\ell(x)$. Defn of $\varepsilon_{\text{mach}}$. Rules of floating point arithmetic (rounding combined with $+ - \times /$)

Sum numbers in magnitude smallest to largest, and why other order is worse.

Catastrophic cancellation, spotting it and predicting its size by chasing epsilons (keeping only dominant ones); using math to rewrite a formula to avoid it.

Relative condition number of a problem $\kappa(x)$, defn. Intuitive consequence for expectation of relative accuracy for evaluation of a function (this is formalized by Bkw Stab Thm below).

3 Week 3

Finite differencing to approximate derivatives. One-sided, centered, and 3-pt stencil. How to get their orders and estimating CC error associated with finite-precision evaluation of the function. (Advanced: optimal choice of h by equating the two sources of error.)

Backwards stability: definition, concept, how to test for it applying rules of floating point to simple functions $f(x)$ or $f(x_1, x_2)$, ie one or two inputs.

Bkw Stab Thm: a bkw stab algorithm's relative err bounded by $\kappa(x)O(\varepsilon_{\text{mach}})$, apply it.

Roots of unity in complex plane, eg solving $z^n = c$ for n integer, $c > 0$ real.

2-norm of a vector, and (induced) norm of a matrix $\|A\|$, definition, understanding in terms of largest growth factor (longest semi-axis of ellipsoid produced when A acts on unit sphere). Formula from HW3: $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$.

Condition number of a matrix $\kappa(A) = \|A\| \cdot \|A^{-1}\|$, and that it is *worst-case* bound of κ for the linear system $A\mathbf{x} = \mathbf{b}$, with respect to input data $\mathbf{b}$.[1]

4 Topics removed since 2013

Stability (as opposed to backward stability). Thus 2013 Midterm 1 6(e) is not relevant for us.

5 Practise questions

Also see worksheets, homeworks, Quiz 1 from 2013 and 2014, and midterm 1 from 2013.

1. Is $\frac{e^n}{10-ne^n} = O(n^{-1})$ as $n \to \infty$? Prove it.
2. Prove if $\log x = O(x)$ as $x \to \infty$? As $x \to 0$?
3. Is $\log n = o(\log(n^2))$ as $n \to \infty$? If so prove it; if not, what else can be said?
4. is $\cos(n)e^{-\sqrt{n}} = O(n^{-10})$ as $n \to \infty$?
5. Write a Newton iteration to solve $x^3 - x = 1$. What function of the error creates a linear graph when plotted vs iteration number n?
6. Use Taylor's theorem to give a simple upper bound on the absolute error in approximating $\cos x$ by $1 - x^2/2$ which applies in $|x| < 0.5$.
7. Fixing any $x > 0$ and $r > 0$, show that the n-term Taylor series for e^x about 0 has error $O(r^n)$. State the type of convergence this implies.
8. Estimate the relative error introduced when a floating point machine evaluates $f(x) = 1 + x$.
9. Write all solutions to $z^3 = 8i$ in the form $re^{i\theta}$.
10. The matrix A turns the vector $(3, 4)$ into the vector $(5, 12)$. Use this to give a bound on $\|A\|$ (upper, lower?) Also use it to give a bound on $\|A^{-1}\|$ (upper, lower?)

[1] for that matter, A too, but we didn't do that

11. Use Taylor's theorem to bound the error of the centered-difference formula for $f'(x)$, ie evaluating at $x \pm h/2$, in exact arithmetic. (This is the "Taylor error".) State the convergence type and order/rate. BONUS: If relative errors in evaluating f are $O(\varepsilon_{\text{mach}})$ what is an optimal choice for h? (Ignore constants size $O(1)$.)

12. Compute the norm of matrix $A = \begin{bmatrix} 1.001 & 1 \\ 1 & 1 \end{bmatrix}$. How many digits of accuracy do you expect in worst-case for linear system involving A? (An exact computation of κ is messy; feel free to make approximations).

13. Bindel–Goodman Ex. 4.6.8.

14. From X-hr: a) Prove if $2N^3 + N^2 = O(N^2)$ as $N \to \infty$ b) Prove if $\sin x \log x = o(x)$ as $x \to \infty$ c) Prove if $10 \sin x = O(x)$ as $x \to 0$ d) Prove if $10 \tan x = O(x)$ as $x \to 0$.

15. Write a Newton iteration to solve $x^3 - x = 1$. What function of the error creates a linear graph when plotted vs iteration number n?

16. What is a rigorous bound on the error of the unsymmetric finite-difference approx $f'(x) \approx \frac{f(x+2h)-f(x-h)}{3h}$? Use exact arithmetic (ignore rounding error). Your bound should hold for all $h > 0$ and may involve properties of f. Then write it in big-O notation.

17. Now accounting for floating point error (assume f evaluated to $\varepsilon_{\text{mach}}$), what is the optimal h to get the best accuracy in the previous question? What roughly is this accuracy?

6 Some practise question answers

1. Yes. To prove use $10/(ne^n) < 2$ for all $n > 1$, so $n_0 = 1$ and $C = 2$.
2. l'Hôpital's rule both times. Ans: Yes, no.
3. No. But it's big-O.
4. Yes, and it's little-o too.
5. $x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1}$. Quadratic convergence, so $\log(\log 1/\varepsilon_n)$ vs n linear.
6. We're using terms 0, 1 (which has zero coefficient), and 2 here, so the thm says error (call ε) is bounded by the next (3rd) term but with x in the derivative replaced by unknown q in the interval. $\varepsilon = f'''(q)(x-0)^3/3!$. So an upper bound over the interval is $|\varepsilon| \le (.5)^3/3! = 1/48$.
7. Taylor theorem, then $\lim_{n\to\infty}(x/r)^n/n! = 0$, so is bounded by a const for all sufficiently large n. Super-exponential convergence.
8. No more than $\frac{2|x|+1}{x+1}\varepsilon_{\text{mach}}$.
9. geom gives $2e^{i\theta}$ where $\theta = \pi/6, 5\pi/6, 9\pi/6$ since each angle when tripled gives $\pi/2$
10. $\|A\| \ge 13/5$. $\|A^{-1}\| \ge 5/13$.
11. abs err upper bnd $(h^2/12)\cdot \max_{q\in(x-h,x+h)} |f'''(q)| = O(h^2)$ ie 2nd-order algebraic convergence. As in lecture. Evaluation error causes $O(\varepsilon_{\text{mach}}/h)$ error in answer. Bonus: balancing this against $O(h^2)$ gives $h = \varepsilon_{\text{mach}}^{1/3}$.
12. norm is about $\sqrt{2}$ (I didn't do exactly). $\kappa(A)$ is about 10^3, so roughly expect 13 digits in solution.
13.
14. no, no, yes, yes. For the trig you can use the leading terms $\sin x = x + O(x^3)$ and $\tan x = x + O(x^3)$ for small x.
15. $x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1}$. Quadratic convergence, so $\log(\log 1/\varepsilon_n)$ vs n linear.
16. absolute error is $O(h)$.
17. $h = O(\varepsilon_{\text{mach}}^{1/2})$, 8 digits.