MATH 56 WORKSHEET : $\sqrt{\ }$ & Quadratic convergence. 4/2/13 Barnett

$$\boxed{\text{Newton iteration} \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \quad (*)$$

A) Pick the simplest $f(x)$ which has a root at $x = \sqrt{y}$ (y is constant data.)
that doesn't involve $\sqrt{\ }$ itself!

Show the Newton iteration gives $x_{n+1} = \frac{1}{2}\left(x_n + \frac{y}{x_n}\right)$:

B) Write Taylor's thm. at $x = z$ (the true root), expanding about x_n, up to $n=1$: [keep $f(x)$ general].

$$f(z) = \ldots$$

Simplify by realizing $f(z)$ is something, divide all by $f'(x_n)$, recognizing x_{n+1} from $(*)$:

You should get $\quad x_{n+1} - z = (\text{stuff}) \cdot (x_n - z)^2$

Finally, make the case that "stuff" tends to a const limit $C = \ldots$?

BONUS (rigorous): what condition on $|x_n - z|$ needed so must move closer to z each iter?

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SOLUTIONS

$$\boxed{\text{Newton iteration} \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \quad (*)$$

"$\sqrt{\ }$" is "square root", don't confuse w/ "root" = zero of a function.

A) Pick the simplest $f(x)$ which has a root at $x = \sqrt{y}$ (y is constant data.)
that doesn't involve $\sqrt{\ }$ itself!

$$f(x) = x^2 - y \qquad (\text{or } y - x^2)$$

$f'(x) = 2x$

Show the Newton iteration gives $x_{n+1} = \frac{1}{2}\left(x_n + \frac{y}{x_n}\right)$:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - y}{2x_n} = \frac{1}{2}\left(x_n + \frac{y}{x_n}\right)$$

B) Write Taylor's thm. at $x = z$ (the true root), expanding about x_n, up to $n=1$:
[keep $f(x)$ general].

$$f(z) = f(x_n) + (z - x_n) f'(x_n) + \frac{(z-x_n)^2}{2!} f''(q) \qquad q \text{ between } x_n \text{ \& } z.$$

Simplify by realizing $f(z)$ is something zero!, divide all by $f'(x_n)$, recognizing x_{n+1} from (*):

so
$$0 = \frac{f(x_n)}{f'(x_n)} + (z - x_n) \cancel{\frac{f'(x_n)}{f'(x_n)}} + \frac{(z-x_n)^2}{2!} \frac{f''(q)}{f'(x_n)}$$

You should get

$$x_{n+1} - z = \left(\underbrace{\text{stuff}}_{\frac{f''(q)}{2f'(x_n)}}\right) \cdot (x_n - z)^2$$

since f', f'' both continuous & $x_n, q \to z$ if converges

Finally, make the case that "stuff" tends to a const limit $C = \frac{f''(z)}{2f'(z)}$?

BONUS (rigorous): what condition on $|x_n - z|$ needed so must move closer to z each iter?

Let $|x_n - z| < \frac{1}{C'}$, where $C' > C$ such that $\frac{\max_{x\in I} |f''(x)|}{2 \min_{x \in I} |f'(x)|} \le C'$ for interval $I \ni x_n$. then $|x_{n+1} - z| \le C'(x_n - z)^2 < |x_n - z|$, closer

[Restrepo p. 72]