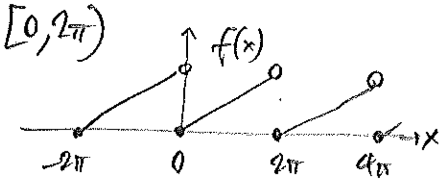
MATH 56 WORKSHEET : Fourier series. Barnett 4/16/13

Recall: $\boxed{f(x) = \sum_{n\in\mathbb{Z}} \hat{f}_n e^{inx}, \text{ where } \hat{f}_m = \frac{1}{2\pi}\int_0^{2\pi} e^{-imx} f(x)\,dx}$

A) Compute $\hat{f}_m$ coefficients for $f(x) = x$ on $[0, 2\pi)$

[Hint: treat $m=0$ separately]



B) Let general f be written as a Fourier series, let's relate $\|f\|^2$ to its coeffs:

$$\|f\|^2 := (f, f) = \left(\sum_n \hat{f}_n e^{inx}, \quad \cdots \quad \right)$$

$= \cdots$

now bring the summation signs outside the inner prod. (assume sums unconditionally convergent...), simplify.

C) If $f \in C([0, 2\pi])$, what does this tell you about the sequence $\hat{f}_n$ as $|n| \to \infty$?

BONUS: redo B) using truncated series $\|f - \sum_{n=-N}^{N} \hat{f}_n e^{inx}\|^2$ & set it ≥ 0.

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⤳ SOLUTIONS

Recall: $$\boxed{f(x) = \sum_{n\in\mathbb{Z}} \hat{f}_n e^{inx}, \text{ where } \hat{f}_m = \frac{1}{2\pi}\int_0^{2\pi} e^{-imx} f(x)\,dx}$$

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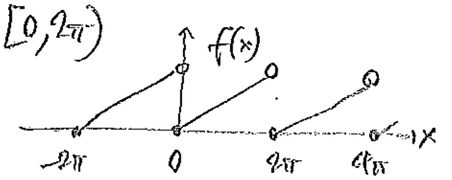

$m=0$: $\hat{f}_0 = \frac{1}{2\pi}\int_0^{2\pi} x\,dx = \frac{1}{2\pi}\cdot\frac{(2\pi)^2}{2} = \pi$ = avg. value.

$m\neq 0$: $\hat{f}_m = \frac{1}{2\pi}\int_0^{2\pi} \underset{\uparrow u}{x}\,\underset{\uparrow v'}{e^{-imx}}\,dx = \frac{1}{2\pi}\left[\frac{x e^{-imx}}{-im}\right]_0^{2\pi} - \frac{1}{2\pi}\left(\frac{1}{-im}\right)\underbrace{\int_0^{2\pi} e^{-imx}\,dx}_{=0 \text{ from class, for } m\neq 0}$

$= \frac{i}{m}$

note $(\alpha f, g) = \alpha^*(f,g)$ ← conj. linear in 1st variable.
$(f, \alpha g) = \alpha(f,g)$

B) Let general f be written as a Fourier series, let's relate $\|f\|^2$ to its coeffs:

note cannot choose n as internal index in sum if want to mix up the sums.

$\|f\|^2 := (f,f) = \left(\sum_n \hat{f}_n e^{inx}, \sum_m \hat{f}_m e^{imx}\right)$ now bring the summation signs outside the inner prod. (assume sums unconditionally convergent...), simplify.

$= \sum_{n,m} \hat{f}_n^* \hat{f}_m \underbrace{(e^{inx}, e^{imx})}_{\text{in class: orthogonality}}$

$= \sum_n \hat{f}_n^* \underbrace{\sum_m \hat{f}_m\, 2\pi\,\delta_{nm}}_{2\pi \hat{f}_n} = 2\pi \sum_n \hat{f}_n^* \hat{f}_n = 2\pi \sum_n |\hat{f}_n|^2.$

C) If $\underbrace{f \in C([0,2\pi])}_{\text{so } \|f\|^2 < \infty \text{, some const.}}$, what does this tell you about the sequence $\hat{f}_n$ as $|n|\to\infty$?

$\sum |\hat{f}_n|^2$ convergent, so $\hat{f}_n \to 0$ as $|n|\to\infty$.

BONUS: redo B) using truncated series $\|f - \sum_{n=-N}^{N} \hat{f}_n e^{inx}\|^2$ & set it ≥ 0. ← see HW4.