MATH 56 WORKSHEET : Conditioning & finite differencing

4/4/13
Barnett

A) Compute relative condition number κ for the problems:

i) $f(x) = x^{\alpha}$ [the input data is x ; α is const.]

ii) $f(x) = 1-x$

When is this ill-conditioned?

iii) $f(x) = \sin x$

When ill-conditioned? [$x \to 0$? $x \to \infty$? other x?]

B) Consider the approximation $f'(x) \approx \dfrac{f(x+h) - f(x)}{h}$, h small

i) Use Taylor expansion about x up to the linear term, and Taylor's Theorem, to bound the error, writing as big-O as $h \to 0$:

ii) If f is evaluated with relative error ε_{mach}, what relative error is induced in $\dfrac{f(x+h)-f(x)}{h}$?

iii) What choice of h balances Taylor and rounding error? [assume $|f| \approx |f'| \approx 1$]

How many digits do you expect?

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SOLUTIONS

A) Compute relative condition number κ for the problems:

i) $f(x) = x^\alpha$ [the input data is x; α is const.]

$f'(x) = \alpha x^{\alpha-1}$

$$\kappa = \left|\frac{x f'(x)}{f(x)}\right| = \left|\frac{\alpha x^\alpha}{x^\alpha}\right| = |\alpha|$$

well-cond unless $|\alpha|$ huge.

ii) $f(x) = 1-x$

$f'(x) = -1$

$$\kappa = \left|\frac{x(-1)}{1-x}\right| = \left|\frac{x}{1-x}\right|$$

$|x| \to \infty$ has $\kappa \to 1$.

When is this ill-conditioned? for $x \approx 1$ (ie $|x-1| < 10^{-3}$, say).

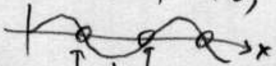
iii) $f(x) = \sin x$

$f'(x) = \cos x$

$$\kappa = \left|\frac{x\cos x}{\sin x}\right| = \left|\frac{x}{\tan x}\right|$$

$x \to 0$ $\kappa \to 1$, well cond.

$x \to n\pi$, $n \neq 0$, $\kappa \to \infty$, ill cond.



When ill-conditioned? [$x \to 0$? $x \to \infty$? other x?]

large $|x|$ in general gives large κ since $\tan x$ rarely v. small.

B) Consider the approximation $f'(x) \approx \dfrac{f(x+h) - f(x)}{h}$, h small

i) Use Taylor expansion about x up to the linear term, and Taylor's Theorem, to bound the error, writing as big-O as $h \to 0$:

$$\frac{1}{h}[f(x+h) - f(x)] = \frac{1}{h}\left[\cancel{f(x)} + hf'(x) + \frac{h^2}{2!}f''(q) - \cancel{f(x)}\right] \quad \text{for some } q \in [x, x+h]$$

$$= f'(x) + \underbrace{\frac{h}{2}f''(q)}$$ as $h \to 0$ this error $= O(h)$

"first order in h"

ii) If f is evaluated with relative error ε_{mach}, what relative error is induced in $\dfrac{f(x+h)-f(x)}{h}$? $\longrightarrow$ machine evaluates

$$\frac{f(x+h)(1+\varepsilon_1) - f(x)(1+\varepsilon_2)}{h}(1+\varepsilon_3) = \frac{f(x+h)-f(x)}{h} + \frac{2|f|\varepsilon}{h}$$

Relative error due to finite precision is $\dfrac{2\varepsilon_{mach}}{h} = O\left(\dfrac{\varepsilon_{mach}}{h}\right)$

iii) What choice of h balances Taylor and rounding error? [assume $|f| \approx |f'| \approx 1$]

set $\dfrac{\varepsilon_{mach}}{h} \approx h \quad \Rightarrow \quad h = \sqrt{\varepsilon_{mach}} \approx 10^{-8}$

How many digits do you expect? error is $O(h)$ ie 10^{-8} ie 8 digits.