MATH 56 WORKSHEET : Asymptotics & Convergence plots

Barnett
3/25/14

A) • Is $\frac{\log n}{n} = O(n^{-1/2})$ as $n \to \infty$? [prove it]

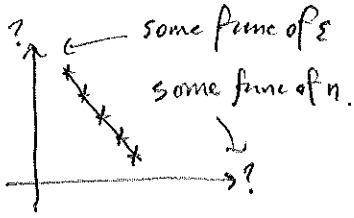
• Say $C > 0$ fixed. Is $C^{-n} = o\left(\frac{1}{n!}\right)$ as $n \to \infty$?

[hint: write out n factors upstairs & downstairs]

• If wanted to write $\frac{10}{n^2} + \frac{3}{n} = O(n^{-p})$ as $n \to \infty$, what's the 'best' p possible (ie smallest bound as $n \to \infty$)?

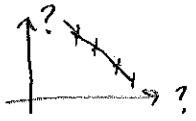
Prove your claim [exhibit a C & n_0]:

B) • Imagine have data pairs (n, ε_n) with algebraic convergence.
What axes should plot so comes out as straight line?



Interpret the slope:

• Do same for exponential convergence:



Interpret the slope.

BONUS: What if ε_n shrinks so fast as to double the # correct digits as $n \to n+1$?
What axes plot now? What is equation for ε vs. n?

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~ SOLUTIONS ~

Barnett 3/25/14

A) • Is $\underbrace{\frac{\log n}{n}}_{f} = O(\underbrace{n^{-1/2}}_{g})$ as $n\to\infty$? [prove it]

$$\frac{f(n)}{g(n)} = \frac{\frac{\log n}{n}}{n^{-1/2}} = \frac{\log n}{n^{1/2}} \xrightarrow{\text{l'H}} \frac{1/n}{\frac{1}{2}n^{-1/2}} = 2\frac{1}{\sqrt{n}} \xrightarrow{\lim n\to\infty} 0$$

so it's little-oh ⇒ it's big-oh too.

Ans: yes.

• Say $C>0$ fixed. Is $\underbrace{C^{-n}}_{f} = o(\underbrace{\frac{1}{n!}}_{g})$ as $n\to\infty$?

[hint: write out n factors upstairs & downstairs]

$$\frac{f}{g} = \frac{C^{-n}}{1/n!} = \frac{n!}{C^n} = \underbrace{\frac{1}{C}\cdot\frac{2}{C}\cdot\frac{3}{C}\cdots\text{"}\frac{C}{C}\text{"}}_{\text{these give some positive const, indep of } n.}\cdots\underbrace{\frac{(n-1)}{C}\cdot\frac{n}{C}}_{\text{these factors all } >1,\ \text{so product grows without limit as } n\to\infty.}$$

Ans: no.

• If wanted to write $\frac{10}{n^2}+\frac{3}{n} = O(n^{-p})$ as $n\to\infty$,

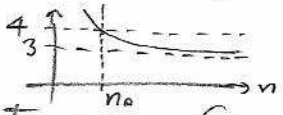
what's the 'best' p possible (ie smallest bound as $n\to\infty$)? ← meaning fastest decay of n^{-p}

Want largest poss. p (fastest decay): $p=1$ gives $\frac{f}{g} = \frac{10}{n}+3$ which is $O(1)$, good.

Prove your claim [exhibit a C & n_0]: Check that $p>1$ gives a growing term eg $3n^{p-1}$

$$\frac{f}{g} = \frac{10}{n}+3 \le 4 = C \quad \text{for } \frac{10}{n}\le 1 \text{ ie } n\ge 10 \text{ ie } n_0 = 9 \text{ (or 10, or anything } >10).$$

sketch:

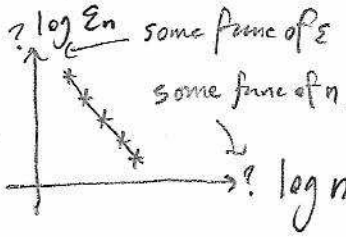


B) • Imagine have data pairs (n, ε_n) with algebraic convergence.

What axes should plot so comes out as straight line?

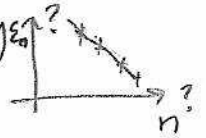
Say $\varepsilon_n = Cn^{-p}$ then $\log\varepsilon_n = \log C - p\log n$ ← like "$y=a+bx$"

Interpret the slope: intercept



• Do same for exponential convergence:

Now $\varepsilon_n = r^n$ so $\log\varepsilon_n = n\log r$



Interpret the slope:

slope $= \log r < 0$

BONUS: What if ε_n shrinks so fast as to double the # correct digits as $n\to n+1$?

$\log\log\frac{1}{\varepsilon_n}$ vs n. ← What axes plot now? What is equation for ε vs. n? $\varepsilon_n = Cr^{(2^n)}$ fun!