How Euler Did It

by Ed Sandifer

Estimating π

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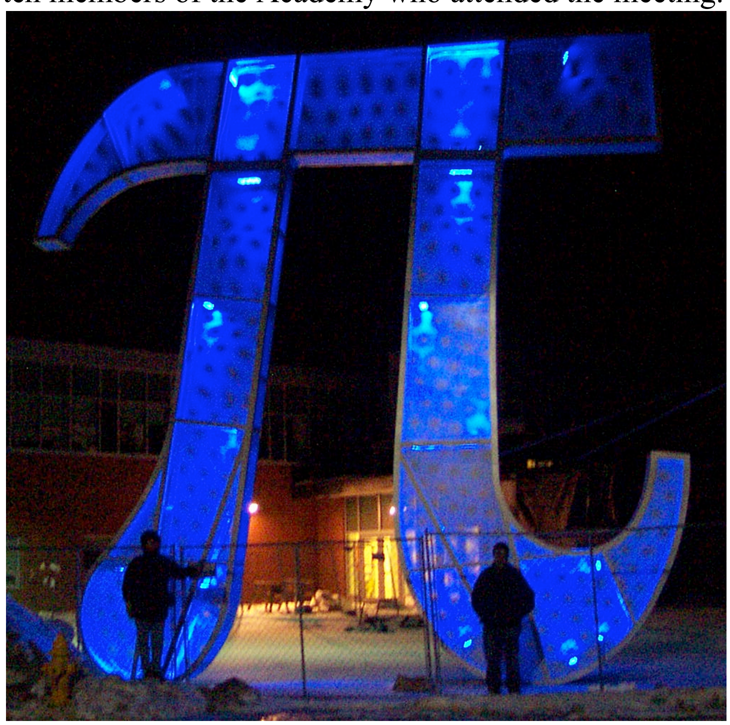
On Friday, June 7, 1779, Leonhard Euler sent a paper [E705] to the regular twice-weekly meeting of the St. Petersburg Academy. Euler, blind and disillusioned with the corruption of Domaschneff, the President of the Academy, seldom attended the meetings himself, so he sent one of his assistants, Nicolas Fuss, to read the paper to the ten members of the Academy who attended the meeting.



The paper bore the cumbersome title "Investigatio quarundam serierum quae ad rationem peripheriae circuli ad diametrum vero proxime definiendam maxime sunt accommodatae" (Investigation of certain series which are designed to approximate the true ratio of the circumference of a circle to its diameter very closely."

Up to this point, Euler had shown relatively little interest in the *value* of π, though he had standardized its notation, using the symbol π to denote the ratio of a circumference to a diameter consistently since 1736, and he found π in a great many places outside circles. In a paper he wrote in 1737, [E74] Euler surveyed the history of calculating the value of π. He mentioned Archimedes, Machin, de Lagny, Leibniz and Sharp. The main result in E74 was to discover a number of arctangent identities along the lines of

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{70} + \arctan\frac{1}{99}$$

and to propose using the Taylor series expansion for the arctangent function, which converges fairly rapidly for small values, to approximate π. Euler also spent some time in that paper finding ways to approximate the logarithms of trigonometric functions, important at the time in navigation tables. The paper ends with the intriguing formula

$$\frac{\sin A}{\cos\frac{1}{2}A\cdot\cos\frac{1}{4}A\cdot\cos\frac{1}{8}A\cdot\cos\frac{1}{16}A\cdot\text{etc.}}=\infty\sin\frac{1}{\infty}A=A.$$

Let us resist the temptation to digress too much further and return to the papers Euler wrote in 1779, one of only 24 papers Euler wrote that year. Most of those papers, including E705, were published in the 1790s, more than ten years after Euler's death.

Euler opens E705 with another history of efforts to approximate π, adding the name Ludolph van Ceulen to his list and noting that Sharp had calculated π to 72 digits, Machin to 100 digits and de Lagny to 128 digits, which Euler describes as a "Herculean task."

To begin his own analysis, Euler reminds us of Leibniz's arctangent series, which gives the angle s in terms of its tangent t as

$$s=t-\frac{1}{3}t^3+\frac{1}{5}t^5-\frac{1}{7}t^7+\frac{1}{9}t^9-\text{etc.}$$

We could take $t = 1$ so that $s=\frac{\pi}{4}$ and get the very-slowly converging approximation

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}-\text{etc.}$$

This is sometimes called the Lebniz series.

Euler also reminds us that he had previously [E74, §12] shown that if $1=\frac{a+b}{ab-1}$, that is to say, if $b=\frac{a+1}{a-1}$ then

$$\arctan 1=\arctan\frac{1}{a}+\arctan\frac{1}{b}$$

This, in turn, is a special case of the formula

$$\arctan\frac{1}{p}=\arctan\frac{1}{p+q}+\arctan\frac{q}{p^2+pq+1}$$

in the case $p = 1$ and $q = a - 1$. [E74, p. 253, §14]

Euler has a whole repertoire of such formulas. Not all of them are mentioned in E74, but they all come easily from the still-more general formula

$$\arctan\alpha=\arctan\beta+\arctan\frac{\alpha-\beta}{1+\alpha\beta}.$$

Without citing any particular formula, Euler proclaims that

$$\arctan\frac{1}{2}+\arctan\frac{1}{3}=\arctan 1=\frac{\pi}{4}.$$

This can be found from the first formula above by taking $a = 2$, so that $b = 3$.

This leads to a double series, because from the arctangent series we have

$$\arctan\frac{1}{2} = \frac{1}{2} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} - \frac{1}{7\cdot 2^7} + \text{etc.}$$

and

$$\arctan\frac{1}{3} = \frac{1}{3} - \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} - \frac{1}{7\cdot 3^7} + \text{etc.}$$

These series decrease "in quadruple ratio", that is to say, each term is less than ¼ the size of the previous term, so it converges much more quickly than the series for $t = 1$.

Note that Euler is using something like the ratio test when he describes this convergence as being in "quadruple ratio." Though this is not the ratio of any two consecutive terms, it is the limit of those ratios. Augustin-Louis Cauchy is usually credited with discovering the ratio test in 1821, 42 years after Euler wrote this paper, but only 23 years after it was published.

We can make the series converge more quickly because the denominators are larger if we know that

$$\arctan\frac{1}{2} = \arctan\frac{1}{3} + \arctan\frac{1}{7}.$$

This follows from the second of Euler's arctangent addition formulas, taking $p = 2$ and $q = 1$. Combining this new fact with the formula for arctan 1, it gives

$$\pi = 4\arctan 1 = 8\arctan\frac{1}{3} + 4\arctan\frac{1}{7}.$$

The problem with this is that the second of the arctangent series requires repeated divisions by 49, and though it converges rather quickly, the computations are difficult.

Euler seeks the best of both worlds, rapid convergence and easy calculations. He lets

$$\begin{aligned} s &= t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \text{etc.} \\ stt &= \quad t^3 - \frac{t^5}{3} + \frac{t^7}{5} - \frac{t^9}{7} + \text{etc.} \end{aligned}$$

so

$$s + sst = t + \frac{2}{3}t^3 - \frac{2}{3\cdot 5}t^5 + \frac{2}{5\cdot 7}t^7 - \text{etc.} = t + s'tt,$$

where this equation defines a new variable s', not to be confused with the derivative of s.

Then

$$s' = \frac{2}{3}t - \frac{2}{3\cdot 5}t^3 + \frac{2}{5\cdot 7}t^5 - \frac{2}{7\cdot 9}t^7 + \text{etc.}$$

so that

$$s'tt = \frac{2}{1\cdot 3}t^3 - \frac{2}{3\cdot 5}t^5 + \frac{2}{5\cdot 7}t^7 - \text{etc.}$$

Likewise, by series expansions, he shows that if s'' is defined by the equation

$$s'(1+tt) = \frac{2}{3}t + s''t$$

then

$$s''tt = \frac{2\cdot 4}{1\cdot 3\cdot 5}t^3 - \frac{2\cdot 4}{3\cdot 5\cdot 7}t^5 + \text{etc.}$$

and so on, defining s''', etc.

Solving for s, s', etc., we get

$$s = \frac{t}{1+tt} + \frac{s'tt}{1+tt}$$
$$s' = \frac{2t}{3(1+tt)} + \frac{s''tt}{1+tt}$$
$$s'' = \frac{2\cdot 4t}{3\cdot 5(1+tt)} + \frac{s'''tt}{1+tt}$$
$$s''' = \frac{2\cdot 4\cdot 6t}{3\cdot 5\cdot 7(1+tt)} + \frac{s''''tt}{1+tt}$$

etc.

Substituting each of these into the one before it gives

$$s = \frac{t}{1+tt} + \frac{2}{3}\cdot\frac{t^3}{(1+tt)^2} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{t^5}{(1+tt)^5} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\frac{t^7}{(1+tt)^7} + \text{etc.},$$

which reduces to

$$s = \frac{t}{1+tt}\left[1 + \frac{2}{3}\left(\frac{tt}{1+tt}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{tt}{1+tt}\right)^2 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\left(\frac{tt}{1+tt}\right)^3 + \text{etc.}\right].$$

This is convenient because each term is the previous term multiplied by $\frac{tt}{1+tt}$ and by a simple fraction of the form $\frac{2n}{2n+1}$.

Euler derives this same formula by a different method that begins by writing the angle s as an integral, $s = \int \frac{dt}{1+tt}$, but we will omit that derivation here.

If we apply Euler's series for s to the identity indicated above, namely

$$\pi = 4\arctan\frac{1}{2} + 4\arctan\frac{1}{3},$$

then for the first part, where $t = \frac{1}{2}$, we get

$$\arctan\frac{1}{2} = \frac{2}{5}\left(1 + \frac{2}{3}\cdot\frac{1}{5} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{1}{5^2} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\frac{1}{5^3} + \text{etc.}\right)$$

and for the second part, where $t = \frac{1}{3}$, we get

$$\arctan\frac{1}{3} = \frac{3}{10}\left(1 + \frac{2}{3}\cdot\frac{1}{10} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{1}{10^2} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\frac{1}{10^3} + \text{etc.}\right)$$

Thus, the value for π can be expressed as the sum of two series,

$$\pi = \left\{\begin{array}{l} \frac{16}{10}\left[1 + \frac{2}{3}\left(\frac{2}{10}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{2}{10}\right)^2 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\left(\frac{2}{10}\right)^3 + \text{etc.}\right] \\ + \frac{12}{10}\left[1 + \frac{2}{3}\left(\frac{1}{10}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{1}{10}\right)^2 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\left(\frac{1}{10}\right)^3 + \text{etc.}\right] \end{array}\right\}$$

Euler tells us that these two series are obviously much less work because the denominators have factors of 10 and because they are "greatly convergent."

The sum of the given terms, up to the third powers, gives

$$\begin{aligned} \pi &\approx 1.853318094 + 1.286948572 \\ &= 3.140266666 \end{aligned}$$

Euler does similar calculations starting with the identity

$$\pi = 8\arctan\frac{1}{3} + 4\arctan\frac{1}{7}.$$

He finds

$$\arctan\frac{1}{7} = \frac{7}{50}\left(1 + \frac{2}{3}\cdot\frac{1}{50} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{1}{50^2} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\frac{1}{50^3} + \text{etc.}\right)$$

so

$$\pi = \left\{\begin{array}{l} \frac{24}{10}\left[1 + \frac{2}{3}\left(\frac{1}{10}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{1}{10}\right)^2 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\left(\frac{1}{10}\right)^3 + \text{etc.}\right] \\ + \frac{28}{50}\left[1 + \frac{2}{3}\left(\frac{2}{100}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{2}{100}\right)^2 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\left(\frac{2}{100}\right)^3 + \text{etc.}\right] \end{array}\right\}.$$

This gives the approximation $\pi \approx 3.141485325$.

In general, each pair of terms in these series adds about one correct decimal place to the approximation.

Euler goes on to lead us through arctangent identities that lead to faster and easier calculations. Because

$$\arctan\frac{1}{3} = \arctan\frac{1}{7} + \arctan\frac{2}{11}.$$

he gets

$$\pi = 12\arctan\frac{1}{7} + 8\arctan\frac{2}{11}.$$

Then from the identity

$$\arctan\frac{2}{11} = \arctan\frac{1}{7} + \arctan\frac{3}{79}$$

he gets

$$\pi = 20\arctan\frac{1}{7} + 8\arctan\frac{3}{79}.$$

Note how the fraction $\frac{1}{7}$ was problematic when he used the Taylor series for the arctangent because it led to fractions involving 49ths. Here, though, it gives 50ths (disguised as $\frac{2}{100}$), and for $t = \frac{3}{79}$, we have

$$\frac{tt}{1+tt} = \frac{9}{6250} = \frac{144}{100\,000}.$$

This gives the convenient series.

$$\arctan\frac{3}{79} = \frac{237}{6250}\left[1 + \frac{2}{3}\left(\frac{144}{100\,000}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{144}{100\,000}\right)^2 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\left(\frac{144}{100\,000}\right)^3 + \text{etc.}\right]$$

These two arctangent approximations lead to Euler's best explicit approximating series of the paper,

$$\pi = \left\{ \begin{array}{l} \frac{28}{10}\left[1 + \frac{2}{3}\left(\frac{2}{100}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{2}{100}\right)^2 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\left(\frac{2}{100}\right)^3 + \text{etc.}\right] \\ + \frac{30336}{100\,000}\left[1 + \frac{2}{3}\left(\frac{144}{100\,000}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{144}{100\,000}\right)^2 + \text{etc.}\right] \end{array} \right\}.$$

Euler calculates the first of these series accurately to 12 decimal places and the second to 17 places, but for some reason he doesn't add them together to give an approximation of π. We will speculate on this mystery at the end of the column.

Eighteen pages into a twenty-page paper, Euler suddenly changes gears and goes back to the original arctangent series,

$$\arctan t = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \frac{1}{9}t^9 - \text{etc.}$$

He denotes by Σ the partial sum of the first n terms of this series, so

$$\Sigma = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \cdots \pm \frac{t^{2n-1}}{2n-1}.$$

Now, changing the meaning of the symbol s from what it had meant earlier in the article, he lets s denote the remainder term, so that

$$s = \frac{t^{2n+1}}{2n+1} - \frac{t^{2n+3}}{2n+3} + \frac{t^{2n+5}}{2n+5} - \text{etc.},$$

where we notice he's being a little sloppy about the ambiguity of the signs.

All this makes

$$\arctan t = \Sigma \pm s$$

Euler parallels his series manipulations from sections 4 and 5 find series for s, for stt, then $s(1+tt)$ to define s' with the relation

$$s(1+tt) = \frac{t^{2n+1}}{2n+1} s'tt.$$

Similarly, he defines s'' with

$$s'(1+tt) = \frac{2t^{2n+1}}{(2n+1)(2n+3)} + s''tt \text{, etc.}$$

Substituting, then factoring, gives

$$\begin{aligned} s &= \frac{2^{2n+1}}{(2n+1)(1+tt)} + \frac{2^{2n+3}}{(2n+1)(2n+3)(1+tt)^2} + \frac{2^{2n+5}}{(2n+1)(2n+3)(2n+5)(1+tt)^3} + \text{etc.} \\ &= \frac{2^{2n+1}}{(2n+1)(1+tt)}\left(1 + \frac{2tt}{(2n+1)(2n+3)} + \frac{2\cdot 4t^4}{(2n+1)(2n+3)(1+tt)^2} + \text{etc.}\right) \end{aligned}$$

Euler does no examples with this series, but he tells us that it is even more convergent than the preceding one because the denominators, with those factors $(2n + 1)$, etc. make the denominators much larger than the numerators.

Just ten days later, Euler sent Nicholas Fuss to the Academy meeting with another paper [E706] on approximating π. This one was titled "De novo genere serierum rationalium et valde convergentium quibus ratio peripheriae ad diametrum exprimi posttest" (On a new kind of strongly convergent rational series which is able to express the ratio of the circumference to the diameter). This paper is much shorter, only five pages in the original, seven pages in the *Opera omnia*,

Euler notes that

$$4 + x^4 = (2 + 2x + xx)(2 - 2x + xx)$$

so

$$\int \frac{(2+2x+xx)\,dx}{4+x^4} = \int \frac{dx}{2-2x+xx}.$$

He denotes this last integral by $\odot$, the astrological symbol for the Sun, so that

$$\odot = \int \frac{dx}{2-2x+xx}.$$

But this last integrates to give an arctangent, so that

$$\odot = \arctan\frac{x}{2-x}$$

where, as so often happens, Euler means us to take the particular antiderivative that is zero when $x = 0$.

Now, Euler returns to the form $\int \frac{(2+2x+xx)\,dx}{4+x^4}$, which he rewrites as

$$\begin{aligned}\int \frac{(2+2x+xx)\,dx}{4+x^4} &= 2\int \frac{dx}{4+x^4} + 2\int \frac{x\,dx}{4+x^4} + \int \frac{xx\,dx}{4+x^4} \\ &= 2A + 2B + C \\ &= \arctan\frac{x}{2-x}\end{aligned}$$

where Euler uses, instead of *A, B* and *C*, the astrological symbols, ♄, ♃ and ♂, which are the symbols for Saturn, Jupiter and Mars, respectively.

Euler expands each of these integrands as series, then integrates to get

$$\begin{aligned}A &= \frac{x}{4}\left[1 - \frac{1}{5}\cdot\frac{x^4}{4} + \frac{1}{9}\left(\frac{x^4}{4}\right)^2 - \frac{1}{13}\left(\frac{x^4}{4}\right)^3 + \text{etc.}\right] \\ B &= \frac{xx}{8}\left[1 - \frac{1}{3}\cdot\frac{x^4}{4} + \frac{1}{5}\left(\frac{x^4}{4}\right)^2 - \frac{1}{7}\left(\frac{x^4}{4}\right)^3 + \text{etc.}\right] \\ C &= \frac{x^3}{4}\left[\frac{1}{3} - \frac{1}{7}\cdot\frac{x^4}{4} + \frac{1}{11}\left(\frac{x^4}{4}\right)^2 - \frac{1}{15}\left(\frac{x^4}{4}\right)^3 + \text{etc.}\right]\end{aligned}$$

Now, using the facts that

$$\pi = 8\arctan\frac{1}{3} + 4\arctan\frac{1}{7}$$

and that

$$\arctan\frac{x}{2-x} = 2A + 2B + C$$

to get his grand result,

$$\pi = \left\{ \begin{array}{l} 2\left(1 - \frac{1}{5}\cdot\frac{1}{64} + \frac{1}{9}\cdot\frac{1}{64^2} - \frac{1}{13}\cdot\frac{1}{64^3} + \text{etc.}\right) \\ +\frac{1}{2}\left(1 - \frac{1}{3}\cdot\frac{1}{64} + \frac{1}{5}\cdot\frac{1}{64^2} - \frac{1}{7}\cdot\frac{1}{64^3} + \text{etc.}\right) \\ +\frac{1}{4}\left(\frac{1}{3} - \frac{1}{7}\cdot\frac{1}{64} + \frac{1}{11}\cdot\frac{1}{64^2} - \frac{1}{15}\cdot\frac{1}{64^3} + \text{etc.}\right) \\ \frac{1}{2}\left(1 - \frac{1}{5}\cdot\frac{1}{1024} + \frac{1}{9}\cdot\frac{1}{1024^2} - \frac{1}{13}\cdot\frac{1}{1024^3} + \text{etc.}\right) \\ +\frac{1}{16}\left(1 - \frac{1}{3}\cdot\frac{1}{1024} + \frac{1}{5}\cdot\frac{1}{1024^2} - \frac{1}{7}\cdot\frac{1}{1024^3} + \text{etc.}\right) \\ +\frac{1}{64}\left(\frac{1}{3} - \frac{1}{7}\cdot\frac{1}{1024} + \frac{1}{11}\cdot\frac{1}{1024^2} - \frac{1}{15}\cdot\frac{1}{1024^3} + \text{etc.}\right) \end{array} \right\}$$

Euler claims that this converges very rapidly and that it is very easy to use because the series only use powers of two. Euler makes no effort to perform the calculations, nor does he try it on any of his other arctangent identities like

$$\pi = 20\arctan\frac{1}{7} + 8\arctan\frac{3}{79}$$

or

$$\pi = 4\arctan 1 = 8\arctan\frac{1}{3} + 4\arctan\frac{1}{7}$$

Thus, over the course of these two papers, Euler has given us instructions on how to calculate π to many decimal places, though he has not undertaken the calculations himself. Indeed, these ideas were "in the air" of the era. The Slovenian mathematician and artilleryman Baron Jurij Vega adapted the ideas in [E74] and calculated π to 140 decimal places, in a book he first published in 1784, [Vega 1835, vol. 2] breaking de Lagny's record of 128 digits. In an effort to be admitted as a foreign member of the St. Petersburg Academy, he sent a synopsis [Vega 1798] to the Academy, which they published in the same issue of their *Nova acta* as Euler's two articles E705 and E706.

We close by speculating why Euler did not finish his calculation in E705, and at the same time why he used such strange symbols in his calculations in E706. In 1779, Euler was blind and in ill health. He relied on his team of assistants, Fuss, Gmelin, his son J. A. Euler and others, to write up the details and to check the calculations. Now, suppose that Euler and his assistants were just about to finish writing E705 as an eighteen-page paper when suddenly Euler had another idea. They might have been too distracted by finishing their calculations to write up the new ideas. That would explain both the unfinished calculations and the abrupt change of gear at the end of the paper.

Then Euler may have continued to think about the problem and came across the idea that he used in E706. Euler didn't send another paper to the Academy until August 12, so perhaps he was ill, or maybe he left early for the July vacation days, and it would be like his students to "play" a bit when the Master was away and use those funny symbols as they wrote up the E706. This is, of course, speculation. Though it is consistent with the times and the personalities involved, we can't know if it is true.

Note: The photograph at the beginning of this column is of a 20-foot tall sculpture of the Greek letter π, planned and built by artist Barbara Grygutis in 2008 and standing outside Henry Abbot Technical High School in Danbury, CT, less than three minutes from Exit 6 on I-84 westbound, five minutes from Exit 5 eastbound. At night, it is illuminated with programmable LED lights.

References:

[E74] De variis modis circuli quadraturam numeris proxime exprimendi, *Commentarii academiae scientiarum Petropolitanae*, **9**, (1737) 1744, pp. 222-236. Reprinted in *Opera omnia*, Series I vol. 14, pp. 245-259. Also available online at EulerArchive.org.

[E705] Investigatio quarundam serierum, quae ad rationem peripheriae circuli ad diametrum vero proxime definiendam maxime sunt accommodatae, *Nova acta academiae scientiarum Petropolitinae* **11** (1793) 1798, pp. 133-149, 167-168, Reprinted in *Opera omnia*, Series I vol . 16, pp 1-20. Also available online at EulerArchive.org.

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[Vega 1798] Vega, Géorge von, Detérmination de la demi-circonférence d'un cercle dont le diameter est =1, exprimée en 140 figures decimals, Supplement to the *Nova Acta academiae scientiarum Petropolitanae* **11** (1793), 1798, p. 41-44.

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How Euler Did It is updated each month.