# Math 56 Compu & Expt Math, Spring 2014: Homework 4

due 10am Thursday April 24th

*Exploring beautiful properties and applications of things Fourier. A bit shorter to let you recover from midterm.*

1. Fourier series theory. As in lecture, let $\|\cdot\|$ be the $L_2$-norm on $(0, 2\pi)$.

   (a) Find a *unit-norm* function orthogonal to $f(x) = x$ on $0 \le x < 2\pi$. [Hint: do what you'd do in vector-land, eg pick some simple new function and project away its $f$ component].

   (b) Combine the Fourier series for the $2\pi$-periodic function defined by $f(x) = x$ for $0 \le x < 2\pi$ that we computed on the worksheet with Parseval's relation to evaluate $\sum_{m=1}^{\infty} m^{-2}$

   (c) Compute the Fourier series for the $2\pi$-periodic function defined by $f(x) = x^2$ in $-\pi \le x < \pi$. [Hint: shift the domain of integration to a convenient one.] Comment on how $\hat{f}_m$ decays for this continuous (but not $C^1$) function compared to the discontinuous function in (b).

   (d) Take a general $f$ written as a Fourier series, and take the derivative of both sides[1]. What then is the $m$th Fourier coefficient of $f'$ ? Prove that if $|f'|$ is bounded, $|\hat{f}_m| = O(1/|m|)$, or better.

   (e) Use the previous idea to prove a bound on the decay of Fourier coefficients when $f$ has $k$ bounded derivatives. What bound follows if $f \in C^\infty$ (arbitrarily smooth)? Can you give this a name?

2. Getting to know your DFT. Use numerical exploration followed by proof (each proof is very quick):

   (a) Produce a color image of the real part of the DFT matrix $F$ for $N = 256$. Explain one of the *curves* seen in the image.

   (b) What is $F^2$ ? [careful: matrix product, also don't forget the 0-indexing]. What does $F^2$ do to a vector? (this should be very simple!) Now, for general $N$, prove your claim [Hint: use $\omega$]

   (c) What then is $F^4$ ? Prove this.

   (d) What are the eigenvalues of $F$? Use your previous result to prove this.

   (e) What is the condition number of $F$? Prove this using a result from lecture.

3. The power of trigonometric interpolation, i.e. using just a few samples of a periodic function to reconstruct the function *everywhere*. (Applications to modeling data, etc.)

   (a) Let's interpolate $f(x) = e^{\sin x}$. For $N = 40$, by using the $1/N$-weighted samples at the nodes $x_j = 2\pi j/N$, $j = 0, \ldots, N-1$, and `fft` to do the DFT, find $\tilde{f}_m$. Plot their magnitudes on a log vertical scale vs $m = 0, \ldots, N-1$. Relate to #1(e). By what $|m|$ have the coefficients decayed to $\varepsilon_{\text{mach}}$ times the largest? (This is the effective band-limit of the function at this tolerance.)

   (b) Using $\tilde{f}_m$ as good approximations to the true Fourier coefficients in $-N/2 < m < N/2$, plot the "intepolant" given by this truncated Fourier series, on the fine grid `0:1e-3:2*pi`. Overlay the samples $Nf_j = f(x_j)$ as blobs. [Hint: Remember the negative $m$'s are wrapped into the upper half of the DFT output. Debug until the interpolant passes through the samples]

   (c) By looping over the above for different $N$, make a labeled semi-log plot of the maximum error (taken over the fine grid) between the interpolant and $f$, vs $N$, for even $N$ between 2 and 40. At what $N$ is convergence to $\varepsilon_{\text{mach}}$ reached? (Pretty amazing, eh?) Relate to (a).

[1] You may assume that you can pass the derivative through the sum.

4. Let's prove that, amongst all trigonometric polynomials of degree at most $N/2$, the $N/2$-truncated Fourier series for $f$ is the *best approximation* to $f$ in the $L_2(0, 2\pi)$ norm.

   (a) Let $g$ be a general trig. poly. which can be written $g = \sum_{|n| \le N/2} (\hat{f}_n + c_n) e^{inx}$ for some coefficient "deviations" $c_n$. Recall $f = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx}$. Write the squared $L_2$-norm of the error which is the difference of $g$ and $f$:

$$\begin{aligned} ||\text{error}||^2 &= ||f - g||^2 \\ &= \left|\left| \sum_{|n|>N/2} \hat{f}_n e^{inx} - \sum_{|n| \le N/2} c_n e^{inx} \right|\right|^2 . \end{aligned}$$

   (b) Using the definition of the norm, expand this expression for $||\text{error}||^2$ (you should obtain four terms, two of which will vanish). Whatever you do, do not multiply any of the big sums out. Can you use this expanded form to bound $||\text{error}||^2$ below by $\left|\left|\sum_{|n|>N/2} \hat{f}_n e^{inx}\right|\right|^2$?

   (c) What does this say about how small the error can get? From this lower bound conclude that the error is minimized precisely when each $c_n = 0$, ie when $g$ is the truncated Fourier series of $f$.

5. A quickie on run-times. Consider the "matvec" problem computing $\mathbf{y} = A\mathbf{x}$ for $A$ an $n$-by-$n$ matrix. Write a little code using random $A$ and $\mathbf{x}$ for $n = 4000$, which measures the runtime of Matlab's native `A*x`, and your own naive double loop to compute the same thing. Express your answers in "flops" (flop per sec), and give the ratio. Marvel at how fast the built-in library is.