Math 56 Compu & Expt Math, Spring 2014: Homework 3

due 10am Thursday April 17th

1. Here you learn how to "roll your own" finite difference formulae. Let's say you have access to f at only x, $x+h$, and $x+2h$, and want a 2nd-order accurate approximation for $f'(x)$. Note that this is at the leftmost point of the three; e.g. at the extreme end of a grid of values.

   (a) We want $f'(x) \approx af(x) + bf(x+h) + cf(x+2h)$. Our goal is to solve for the coefficients a, b, c, by setting up a 3-by-3 linear system for them, as follows. Expand the right-hand side via Taylor series about x (you don't need the rigorous remainder term). Since you want this to hold for all functions, you may extract the coefficients in front of $f(x)$ to give one equation, the coefficients of $f'(x)$ to give another, and $f''(x)$ for the third.

   (b) Solve the system either by hand or computer, hence write your new finite difference formula. How do you know the solution is unique?

   (c) Give a *rigorous* upper bound on the error of this formula (in exact arithmetic, i.e. ignore rounding).

2. Stability.

   (a) Show whether subtraction $x_1 - x_2$ is backwards stable (with respect to the two input data) under the rules of floating point.

   (b) Cosine, like anything else done by machine, cannot be more accurate than a relative error of $\varepsilon_{\text{mach}}$ in its output (ie "forward error"). Explain if a machine implementation of cos could be backwards stable near $x = 0$? Near $\pi/2$?

3. Here's a new formula for matrix 2-norm:

$$\|A\| = \sqrt{\lambda_{\max}(A^T A)}, \qquad \text{where } \lambda_{\max}(A^T A) \text{ is the largest eigenvalue of the matrix } A^T A.$$

   (a) Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$. Use the new formula to compute by hand $\|A\|$. How does it compare to the size of the largest eigenvalue of A? (for which you can use `eig`)

   (b) Use this to compute the matrix condition number $\kappa(A)$. Is it well-conditioned?

   (c) Take 100 points $\mathbf{x} \in \mathbb{R}^2$ equi-spaced on the unit circle, and plot them, and $A\mathbf{x}$ for each. What geometric property does $\kappa(A)$ measure of the ellipse produced?

4. Download the two 100×100 matrices `A1` and `A2` from the HW page, and use `textread` to read them into Matlab (you will need to `reshape` them).

   (a) Compare their matrix 2-norms and condition numbers. What worst-case relative errors do you expect for solving linear systems with matrix `A1`? With `A2`? (Use our backward stability theorem, and assume standard double precision.)

   (b) Let's focus on $A =$ `A1`, and load in the RHS $\mathbf{b} =$ `bvec` from the HW page. Solve $A\mathbf{x} = \mathbf{b}$. Then perturb $\mathbf{b}$ by a random vector of norm $\varepsilon_{\text{mach}}$ to get $\tilde{\mathbf{b}}$ (this emulates rounding error applied to the RHS), and solve again $A\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$. What relative norm change $\|\tilde{\mathbf{x}} - \mathbf{x}\|/\|\mathbf{x}\|$ results? Does this match your prediction from (a)?

   (c) Repeat (b) except using the RHS $\mathbf{c} =$ `cvec` from the HW page. Surprising? Is it consistent with (a)? Repeat for random unit-norm RHS vectors—do they behave more like $\mathbf{b}$ or like $\mathbf{c}$?

BONUS Explain the different behaviors [hint: $\|\mathbf{x}\|$], deducing how the directions of $\mathbf{b}$ and $\mathbf{c}$ relate to long and short axes of the ellipse of the image of the unit sphere under A.

(d) Given $A \in \mathbb{R}^{M\times P}$ and $B \in \mathbb{R}^{P\times N}$, prove a bound on $\|AB\|$ in terms of the norms of the individual matrices. [Hint: HW2 6(c).]

5. (a) [Trefethen and Bau, Ex. 13.3] On the same axis, plot $p(x) = (x-2)^9$ around its root two ways. First plot it by evaluating $p(x)$ via its factored form, then plot it via its expanded form $p(x) = x^9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 - 4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512$. Evaluate it at the points $x = 1.920, 1.921, 1.922, ..., 2.08$. Explain the discrepancy.

   (b) Compute the roots of $q(x) = x^2 - \frac{100000001}{10000}x + 1$ using the quadratic formula (in `Matlab`). Notice that $q(x) = (x - 10000)(x - \frac{1}{10000})$. What is the relative error of the computed roots? Explain why one root is good to only 9 digits of accuracy. Can you find a way to get this root to close to machine accuracy using only the input coefficients a, b and c and the other root?