Barnett
4/17/14

— SOLUTIONS —

Math 56 Compu & Expt Math, Spring 2014: Midterm 1

4/17/14, pencil and paper, 2 hrs, 50 points. Good luck!

1. [10 points]

   You wish to use a truncated Taylor series about the origin to approximate the smooth function inverse tangent (arctan) of a given real x, ie,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty} \left(\sin\frac{n\pi}{2}\right)\frac{x^n}{n}$$

4 (a) For a general real x, find a rigorous bound on the error ε_n due to using terms with powers only up to n. [Hint: you don't need Taylor's theorem.] Write it in big-O notation:

$$\varepsilon_n = \tan^{-1}x - \sum_{k=1}^{n}\left(\sin\frac{k\pi}{2}\right)\frac{x^k}{k}$$

just tail of sum survives

$$= \sum_{k>n}^{\infty}\left(\sin\frac{k\pi}{2}\right)\frac{x^k}{k}$$

so $$|\varepsilon_n| \le \sum_{k>n}^{\infty}\left|\sin\frac{k\pi}{2}\right|\frac{|x|^k}{k}$$

use $|\sin| \le 1$. & $\frac{1}{k} \le 1$ (quite crude)

$$\le \sum_{k>n}^{\infty}|x|^k = |x|^{n+1}\underbrace{\sum_{k=0}^{\infty}|x|^k}_{\text{geom series } = \frac{1}{1-|x|}}$$

$$= C|x|^n = O(|x|^n).$$

[... in fact it's $o(|x|^n)$]

3 (b) What type and rate or order of convergence is this? (Your answer may depend on x). Are there any x where it doesn't converge?

it certainly does.

Exponential convergence w/ rate $r = |x|$

-1 $+1$

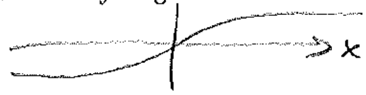
r^n diverges for $r > 1$, ie $x < -1$ & $x > +1$ doesn't converge.

2 (c) Estimate the minimum n needed to get an absolute error of 10^{-16} at $x = 0.3$:

$r = |x| = 0.3$ so want $r^n \approx 10^{-16}$ assuming const $C \approx 1$

$r^2 = 0.09 \approx 10^{-1}$ so need about $n = 32$ (16 nonzero terms).

1 (d) What can you deduce about locations of any singularities of $\tan^{-1}$?

Well, it's smooth 

so can't be on $\mathbb{R}$.

The nearest singularity must be distance $R = 1$ from the origin (in $\mathbb{C}$). since the rate $r = \frac{|x - x_0|}{R}$ in general

1 BONUS: use Euler's relation to find the singularity/singularities

We guess that $\tan^{-1} z$ blows up at singularities (may not be true).

ie $\tan^{-1} z = \pm\infty$ so $z = \tan(\pm\infty)$ but we know not real, so try ∞ in different direction

Euler: $\tan\theta = \frac{1}{i}\frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}} = \frac{1}{i}\frac{1 - e^{-2i\theta}}{1 + e^{-2i\theta}} \to \pm i$, setting $\theta \to \infty$ in any nonreal direction.

2. [8 points] As in homework we wish to find the root of the function $\sin x$ lying in $[3, 4]$.

2 (a) Write down the Newton iteration for this function:

$f(x) = \sin x$, $f'(x) = \cos x$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \tan x_n$$

4 (b) Sketch the proof that the Newton iteration for a general function $f(x)$ is quadratically convergent to its root $f(z) = 0$: [Hint: evaluate at z the Taylor expansion about x_n]

vigorous from of remainder

$$0 \overset{\text{since root}}{=} f(z) = f(x_n) + (z - x_n) f'(x_n) + \frac{(z - x_n)^2}{2!} f''(q)$$

q btw. z & x_0

Divide by $f'(x_n)$:

$$0 = \underbrace{\frac{f(x_n)}{f'(x_n)}}_{x_n - x_{n+1} \text{ by the iteration.}} + (z - x_n) + (z - x_n)^2 \frac{f''(q)}{2 f'(x_n)}$$

$$\text{So} \quad x_{n+1} - z = \underbrace{\frac{f''(q)}{2 f'(x_n)}}_{\leq C} (x_n - z)^2$$

$\leq C$, some const, for all q near the root z.

Let $\varepsilon_n := x_n - z$, then $\varepsilon_{n+1} \leq C \varepsilon_n^2$ Quadratic conv.

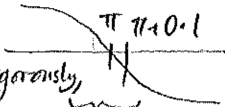
z $\quad \varepsilon_0 = 10^{-1}$

2 (c) Say for $f(x) = \sin x$ the iteration starts with $x_0 = \pi + 0.1$. Use the above to find the precise number of iterations required to reach the machine relative error of 10^{-16}.

$C = \dfrac{f''(q)}{2f'(x_n)}$ ← $f''(q) = -\sin q$ so bounded by 1 in size

← x_n very close to $z = \pi$, where $f'(\pi) = 1$.

So, in interval $[\pi - 0.1, \pi + 0.1]$, can choose $C = 1$



rigorously, we have to know $\cos x > 1/2$ here.

Thus $\varepsilon_1 \leq 1 \cdot \varepsilon_0^2 = 10^{-2}$, $\quad \varepsilon_2 \leq 1 \cdot \varepsilon_1^2 = 10^{-4}$

etc.. $\varepsilon_3 \leq 10^{-8}$, $\varepsilon_4 \leq 10^{-16}$ $\quad$ We are done in 4 iterations.

3. [6 points] An engineer gives you the following two facts about a matrix A:

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 10^5 \end{bmatrix}, \qquad \text{and} \qquad \min_{\|\mathbf{y}\|=1} \|A\mathbf{y}\| = 10^{-4}.$$

What can you deduce about the worst-case relative error in the approximate solution to $A\mathbf{x} = \mathbf{b}$, as solved by a backward stable algorithm with the usual $\varepsilon_{\text{mach}} \approx 10^{-16}$?

an instance of a growth factor, for this $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\|Ax\| = \left\| \begin{bmatrix} 0 \\ 10^5 \end{bmatrix} \right\| = 10^5$

So $\|A\| \geq 10^5$

$\min_{\|y\|=1} \|Ay\|$ says minimum growth of 10^{-4}, & there is a $\hat{y}$ where achieved,

and a $\underbrace{z = A\hat{y}}_{\text{norm } 10^{-4}}$; so $\underbrace{\hat{y}}_{\text{norm } 1} = A^{-1}z$ $\quad$ so $\|A^{-1}\| = 10^4$

$\kappa(A) = \|A\| \|A^{-1}\| \geq 10^{5+4} = 10^9$ $\quad$ gives worst-case $\kappa(b)$ over all right-hand sides b

Backw Stab. Thm says:

relative err $\varepsilon = O(\kappa \, \varepsilon_{\text{mach}}) \geq 10^9 \cdot 10^{-16} = 10^{-7}$

$\Rightarrow$ Worst-case expect at most 7 digits of accuracy.

BONUS: Explain whether it is possible for certain right hand sides (which?) to guarantee relative error close to $\varepsilon_{\text{mach}}$?

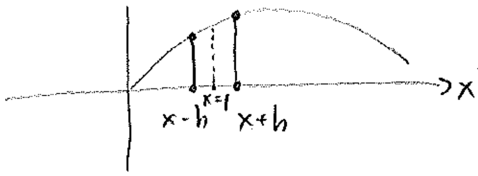
recall $\kappa(b) = \|A^{-1}\| \cdot \boxed{\dfrac{\|b\|}{\|A^{-1}b\|}}$ — we want best (smallest) here, ie A^{-1} to grow b the most, so $\kappa(b) = \|A^{-1}\| \cdot \dfrac{1}{\|A^{-1}\|} = 1$.

This is achieved by $b = z = A\hat{y}$ above.

(or any nonzero multiple)

4. [5 points]

You use a centered-difference approximation to the derivative of $f(x) = \sin x$, at $x = 1$, i.e. using evaluations at $1 - h$ and $1 + h$. Derive a simple rigorous bound on your absolute error in terms of h, including the explicit constant.



note, $f'(1) = \cos 1$ not needed to do this.

remainders

formula $\frac{1}{2h}\left(f(x+h) - f(x-h)\right) = \frac{1}{2h}\Big[\cancel{f(x)} + hf'(x) + \cancel{\frac{h^2}{2!}f''(x)} + \frac{h^3}{3!}f'''(q)$

$\qquad - \cancel{f(x)} - (-h)f'(x) - \cancel{\frac{h^2}{2!}f''(x)} - \frac{(-h)^3}{3!}f'''(q')\Big]$

cancel $\uparrow$ cancel.

gives $f'(x)$, the true derivative.

$\Rightarrow$ absolute error $= \frac{1}{2h}\cdot\frac{h^3}{3!}\underbrace{\left(f'''(q) + f'''(q')\right)}$ for some $q, q' \in [x-h, x+h]$

now $f(x) = \sin x$ so $f'''(x) = -\cos x$, bnded by 1 in size.

$\Rightarrow$ err $\leq \frac{1}{2h}\frac{h^3}{3!}\cdot 2 = \frac{h^2}{6}$ $\leftarrow$ this what 'explicit const' means.

I didn't require you to write to get full score.

BONUS: Around what h would have the smallest error in practice and why?

For small h, roundoff error is of size $O\left(\frac{\varepsilon_{mach}}{h}\right)$, see lecture.

Balancing this against $O(h^2)$ gives $h^2 \approx \frac{\varepsilon_{mach}}{h}$ ie $h \approx \varepsilon_{mach}^{1/3} \approx 10^{-5}$.

5. [10 points]

3 (a) Is the machine evaluation of $f(x) = 1/x$ backwards stable ? (Assume the usual rules of floating-point, assume 1 is stored exactly, and show work.)

$\hat{f}(x) = 1 \oslash fl(x) = \frac{1}{x(1+\varepsilon_1)}(1+\varepsilon_2)$ for some $|\varepsilon_1|, |\varepsilon_2| \leq \varepsilon_{mach}$.

$\stackrel{?\ BS}{=} f(x(1+\varepsilon)) = \frac{1}{x(1+\varepsilon)}$ for any nonzero x, we see

$\varepsilon = \varepsilon_1 + \varepsilon_2 + \cancel{O(\varepsilon^2)}$

$|\varepsilon| \leq 2\varepsilon_{mach}$

so yes, $\varepsilon = O(\varepsilon_{mach})$

It's B.S.

i.e. forward error. we already did in (a), can reuse.

3 (b) Estimate the *relative* error in the machine computation of $f(x) = 1 - 1/x$ for general x (again, assume 1 is stored exactly).

$$\hat{f}(x) = 1 \ominus \left(1 \oslash fl(x)\right)$$

$$= \left(1 - \frac{1}{x(1+\varepsilon_1)}(1+\varepsilon_2)\right)(1+\varepsilon_3) \qquad \text{use } \frac{1}{1+\varepsilon_1} \approx 1 - \varepsilon_1 + O(\varepsilon_1^2)$$

$$= \left[1 - \frac{1}{x}\left(1 - \varepsilon_1 + \varepsilon_2 + \cancel{O(\varepsilon^2)}\right)\right](1+\varepsilon_3)$$

$$= \underbrace{1 - \frac{1}{x}}_{\text{true } f(x)} + \underbrace{\frac{\varepsilon_1 - \varepsilon_2}{x} + \left(1 - \frac{1}{x}\right)\varepsilon_3}_{\text{abs. error}} + \cancel{O(\varepsilon^2)}$$

$$\varepsilon_r = \text{Relative error} = \left|\frac{\text{abs. error}}{f(x)}\right| = \left|\frac{\frac{\varepsilon_1 - \varepsilon_2}{x} + \left(1-\frac{1}{x}\right)\varepsilon_3}{1 - 1/x}\right| = \left|\frac{\varepsilon_1 - \varepsilon_2}{x-1} + \varepsilon_3\right| \leq \left|\frac{2}{|x-1|} + 1\right| \varepsilon_{mach}$$

1 (c) For what range of x is this relative error greater than 1?

Don't have to be exact here.

$$\left|\underbrace{\frac{2}{|x-1|}}_{\text{must exceed} \approx 10^{16}} + 1\right| \varepsilon_{mach} > 1 \qquad \text{so } x \in \left[1 - 2\times 10^{-16}, 1 + 2\times 10^{-16}\right] \text{ will do.}$$

3 (d) It is easy to show that the machine evaluation of $f(x) = 1 - 1/x$ is backward stable for *all* x not too large (eg $|x| < 10^3$). Explain, citing a relevant theorem, how this is can be true even though you discovered above that some inputs cause disastrous relative error. [Hint: a picture may help]

→ due to catastrophic cancellation, by the way.

If backward stable, we can apply:

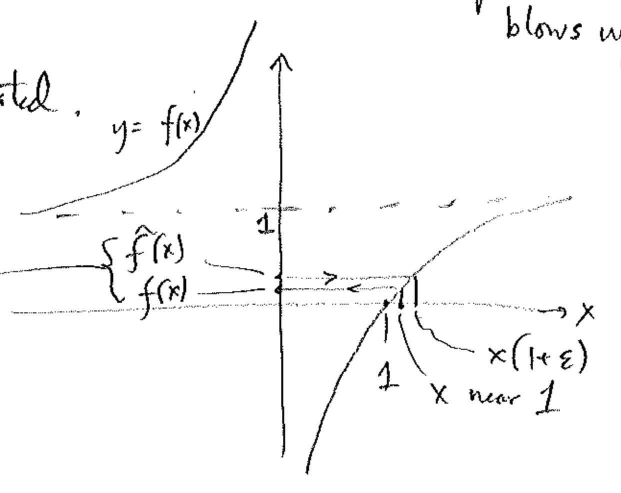
Bkw. Stab. Thm, says $\varepsilon_r = O(\kappa(x)\,\varepsilon_{mach})$ — relative (forward) error.

$$\kappa(x) = \left|\frac{f'(x)\,x}{f(x)}\right| = \left|\frac{-1/x^2 \cdot x}{1 - 1/x}\right| = \frac{1}{|x-1|} \to \infty$$

at the same place that ε_r blows up,

so the theorem isn't violated.

large relative error here.

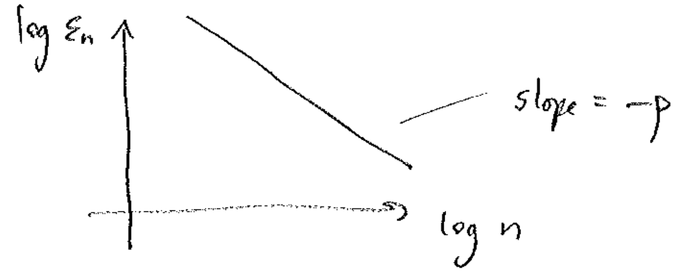


$\varepsilon = O(\varepsilon_{mach})$ backward error small.

6. [11 points] Short-answer questions.

2 (a) What axes would you plot to show a linear graph for error ε_n converging algebraically with n?

$\varepsilon_n = n^{-p}$



2 (b) Prove whether $n = O(\frac{\sqrt{n+n^2}}{\log n})$ as $n \to \infty$.

$f(n)$ $g(n)$

$$\left|\frac{f(n)}{g(n)}\right| = \frac{n \log n}{\sqrt{n+n^2}} = \frac{\log n}{\sqrt{1/n+1}} \to \text{no upper bound.}$$

since $\frac{1}{\sqrt{1/n+1}} \geq \frac{1}{\sqrt{2}}$ for all $n \geq 1$, say, there's no C s.t. $\left|\frac{f(n)}{g(n)}\right| \leq C$ $\forall n > n_0$

$\Rightarrow$ no.

2 (c) Prove whether $e^{-n} = o(n^{-2})$ as $n \to \infty$.

$f(n)$ $g(n)$

both numerator & denom $\to \infty$ as $n \to \infty$.
$\Rightarrow$ l'Hôpital's rule

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{e^{-n}}{n^{-2}} = \frac{n^2}{e^n} \longrightarrow \frac{2n}{e^n} \xrightarrow{\text{again}} \frac{2!}{e^n} \xrightarrow{\lim_{n\to\infty}} 0$$

so, yes

i.e, "to what"

3 (d) How does the machine evaluate $(1 + 10^{-16}) + 10^{-16}$? How could you do this addition of three numbers more accurately on this same machine? [Recall $\varepsilon_{\text{mach}} \approx 1.1 \times 10^{-16}$.]

$1 \oplus 10^{-16}$ rounded down to 1, then same happens again $\Rightarrow$ gives 1.

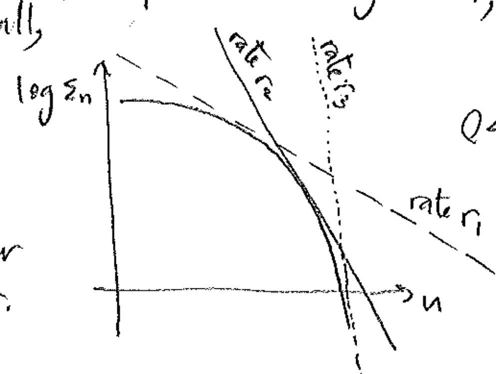
More accurate is to add smallest first, ie $10^{-16} \oplus 10^{-16} \approx 2\times10^{-16}$ to high relative accuracy

then $1 \oplus 2\times10^{-16}$ rounds up to $1 + 2.2\times10^{-16}$ the next floating point number above 1.

2 (e) Give a definition of an error ε_n having super-exponential convergence as $n \to \infty$.

means, exponential at any rate r, no matter how small.

For any $r>0$, no matter how small,
$\varepsilon_n = O(r^n)$



$0 < r_3 < r_2 < r_1$ here.

the graph of ε_n curves downwards to ever-steeper slopes on log-linear plot.