Barnett
5/14/13.

— SOLUTIONS —

Math 56 Compu & Expt Math, Spring 2013: Midterm 2

5/14/13, pencil and paper, 2 hrs, 50 points. Show working. Good luck!

1. [8 points] Consider $f(x)$ a 2π-periodic bounded function with Fourier coefficients $\hat{f}_m$.

[3] (a) Assuming $f(x)$ is real-valued, prove that $\hat{f}_{-m} = (\hat{f}_m)^*$ holds for any integer m.

projection formula $\hat{f}_m = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-imx}\, dx$

so $(\hat{f}_m)^* = \frac{1}{2\pi}\int_0^{2\pi} f^*(x)(e^{-imx})^*\, dx = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{imx}\, dx$

$f^* = f$ since f real

which $= \hat{f}_{-m}$ $\square$

[4] (b) Derive the kth Fourier coefficient of the function $[f(x)]^2$, in terms of the coefficients $\hat{f}_m$.

[This was hard, required insight.. Parseval was a distraction].

note we can't reuse index n since we'll mix the sums.

$$f(x)^2 = \left(\sum_{n\in\mathbb{Z}} \hat{f}_n e^{inx}\right)^2 = \sum_{n\in\mathbb{Z}} \hat{f}_n e^{inx} \cdot \sum_{m\in\mathbb{Z}} \hat{f}_m e^{imx}$$

$$= \sum_{n\in\mathbb{Z}} \hat{f}_n \underbrace{\sum_{m\in\mathbb{Z}} \hat{f}_m e^{i(n+m)x}}$$

this gives e^{ikx} only when $n+m=k$, ie $m = k-n$.

By orthogonality to get the k^{th} Fourier coeff. of f^2, we look for all contributions of the form e^{ikx}

$$= \sum_{n\in\mathbb{Z}} \hat{f}_n \hat{f}_{k-n}$$

←·· looks familiar!

[1] (c) Recognize your previous result as an operation (which one?) applied to the discrete set $\{\hat{f}_m\}_{m\in\mathbb{Z}}$ resulting in the set of Fourier coefficients of f^2.

Let $g = f^2$. Set of coeffs $\{\hat{g}_k\}$ given by acyclic convolution of $\{\hat{f}_m\}$ with itself (A new result!)

BONUS If f is even, $f(-x) = f(x)$ for all x, what is the consequence for the Fourier coefficients?

$$2\pi\hat{f}_m = \int_0^{2\pi} f(x)\, e^{-imx}\,dx \overset{\substack{\text{change var}\\ y=-x}}{=} \int_{-2\pi}^{0} \underbrace{f(-y)\, e^{imy}\,dy}_{\text{evenness}} \overset{\text{periodicity}}{=} \int_0^{2\pi} f(y)\, e^{imy}\,dy = 2\pi\hat{f}_{-m}$$

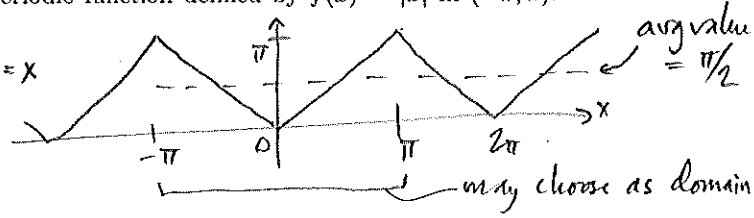
Symmetry of coeffs; combined with f being real would imply coeffs purely real.

[5] 2. [10 points]

(a) Compute the Fourier coefficients of the 2π-periodic function defined by $f(x) = |x|$ in $(-\pi, \pi)$. [Hint: a sketch may help]

$f(x) = -x$ $\quad f(x) = x$

Split integrals into $(-\pi, 0)$ & $(0, \pi)$:



$$\int_0^{\pi} x e^{-imx}\,dx = \frac{i}{m}\Big[x e^{-imx}\Big]_0^{\pi} - \frac{i}{m}\underbrace{\int_0^{\pi} e^{-imx}\,dx}_{\to\ \frac{i}{m}\left[e^{-imx}\right]_0^{\pi}\ =\ \begin{cases} -\frac{2i}{m}, & m \text{ odd} \\ 0, & m \text{ even},\ m\neq 0 \end{cases}}$$

$$= \begin{cases} \frac{\pi i}{m}(-1)^m - \frac{2}{m^2} & m \text{ odd} \\ \frac{\pi i}{m}(-1)^m & m \text{ even},\ m\neq 0 \end{cases}$$

likewise $\displaystyle\int_{-\pi}^{0} x e^{-imx}\,dx = \int_0^{\pi} x e^{imx}\,dx$ = conjugate of above.

Add the two domains: $\displaystyle\int_{-\pi}^{\pi} |x| e^{-imx}\,dx = \begin{cases} -\frac{4}{m^2} & m \text{ odd} \\ 0 & m \text{ even},\ m\neq 0 \end{cases}$

Special case $m=0$: average value $= \pi/2$.

$$\Rightarrow \hat{f}_m = \begin{cases} \pi/2 & m = 0 \\ -\frac{2}{\pi m^2} & m \text{ odd} \\ 0 & m \text{ even},\ m \neq 0 \end{cases}$$

[2] (b) Derive a useful bound on the maximum error of approximating the above function f using N-point trigonometric polynomial interpolation, and state its type and order/rate. [Hint: you should get error vanishing as $N \to \infty$. If you cannot, recheck (a).]

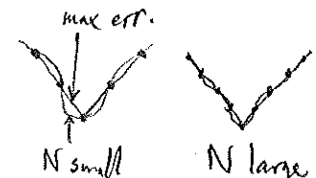
We know max error of interpolation $\leq \overbrace{2 \sum_{|n| \geq N/2} |\hat{f}_n|}^{\text{call } E_N}$ from lecture. (expected to remember this).

$$\text{Then } E_N = 4 \sum_{\substack{n \geq N/2 \\ n \text{ odd}}} \frac{2}{\pi n^2} \leq \frac{8}{\pi} \sum_{n \geq N/2} \frac{1}{n^2}$$

technique from lecture 1 on bounding algebraic sums.

$$\leq \frac{8}{\pi} \int_{N/2}^{\infty} \frac{1}{x^2}\,dx = \frac{8}{\pi} \left[\frac{-1}{x} \right]_{N/2}^{\infty} = \frac{16}{\pi N}$$

$= O(N^{-1})$ 1st order algebraic convergence.



[3] (c) Now say trigonometric polynomial interpolation with $N = 8$ points is performed on the function $f(x) = e^{-3ix}$. Give the vector resulting from the discrete Fourier transform (DFT) of the sample vector:

nodes $x_j = \frac{2\pi j}{N}$ $j = 0, \dots 7$ so $f_j = \frac{1}{N} e^{-\frac{3i \cdot 2\pi j}{N}}$
$= \frac{1}{N} \omega^{-3j}$

By aliasing this appears in $m=5$ DFT entry,

$$\tilde{f} = \begin{array}{c} \begin{matrix} m=0 & m=1 & \cdots & & & m=5 & & \end{matrix} \\ \left[\underbrace{\begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix}}_{\text{positive freqs.}} \; \underbrace{\begin{matrix} 1 & 0 & 0 \end{matrix}}_{\text{negative freqs.}} \right] \end{array}$$

using standard ordering of DFT outputs.

Finally, what interpolant function is produced, and what it its $L^2(0, 2\pi)$ error?

note negative freqs do appear

$$\text{Interpolant} = \sum_{|m| < N/2} \tilde{f}_m e^{imx}$$

where $\tilde{f}_{-3} = \tilde{f}_5 = \tilde{f}_{13} = \cdots$ etc by periodicity of definition of DFT.

$$= e^{-3ix}$$

It is exact, ie has no error.

sorry; HW5 had $\sin^3 x$

3. [6 points] Consider the 2π-periodic function $f(x) = |\sin^5 x|$ ~~from homework~~, which is C^4 continuous.

[2] (a) What can you say about the decay of its Fourier coefficients? (You may state a result without proof.)

We derived (in homework) $f \in C^k \Rightarrow \hat{f}_m = O(|m|^{-k})$

or even better $o(|m|^{-k})$. $k=4$, so $\hat{f}_m = O(|m|^{-4})$

[4] (b) Find a bound on the absolute *error* in the zeroth Fourier coefficient due to approximating it by the zeroth component of the DFT of f sampled on a regular N-point grid.

↳ meaning we want the error $\tilde{\hat{f}}_0 - \hat{f}_0$

Use aliasing formula $\tilde{f}_m = \cdots + \hat{f}_{m-N} + \hat{f}_m + \hat{f}_{m+N} + \cdots$

Set $m=0$: $\tilde{\hat{f}}_0 - \hat{f}_0 = \hat{f}_N + \hat{f}_{2N} + \cdots + \hat{f}_{-N} + \hat{f}_{-2N} + \cdots$

$\le \frac{C}{N^4} + \frac{C}{(2N^4)} + \cdots + \frac{C}{N^4} + \frac{C}{(2N)^4} + \cdots$ ← when N sufficiently large

$\le \Big(\underbrace{2C\sum_{k=1}^{\infty}\frac{1}{k^4}}_{\text{some const.}}\Big)\cdot\frac{1}{N^4} = O(N^{-4})$

[BONUS] Show that (b) gives a bound on the error of a quadrature scheme for $\int_0^{2\pi} f(x)dx$.

$2\pi\hat{f}_0 = \int_0^{2\pi} f(x)dx$ by projection formula, $2\pi\tilde{f}_0 = \frac{2\pi}{N}\sum_{j=0}^{N-1} f(x_j)$ ← nodes.

so $\left| \frac{2\pi}{N}\sum_{j=0}^{N-1} f\left(\frac{2\pi j}{N}\right) - \int_0^{2\pi} f(x)dx \right| = O(N^{-4})$

← periodic trapezoid quadrature rule.

4. [10 points]

[3] (a) Find the $N = 4$ periodic convolution of $[1\,2\,3\,0]$ (f) and $[0\,1\,1\,1]$ (g).

$[0\ 1\ 2\ 3]$

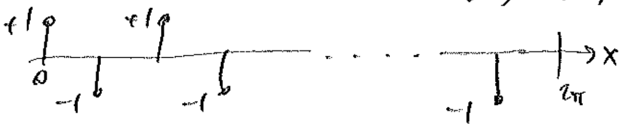
$[3\ 0\ 1\ 2]$ — f shifted cyclically by nonzero entry locations in g.

sum up. $+\ [2\ 3\ 0\ 1]$

$[5\ 4\ 3\ 6]$ same length since periodic

It was also possible to use Danielson–Lanczos lemma here!

[2] (b) Let $N > 0$ be even. What is the DFT of the length-N vector $[1, -1, 1, -1, \dots, -1]$? $= (-1)^j \quad j = 0, \dots N-1$

This is an intuitive one: 

This is the most oscillatory function on the N-point grid, ie at Nyquist freq. So all coeffs zero apart from $m = N/2$. Size of $\tilde{f}_{N/2}$ get from DFT: $\tilde{f}_{N/2} = \sum_{j=0}^{N-1} (\omega^{N/2})^{-j} f_j = \sum_{j=0}^{N-1} (-1)^j (-1)^j = N$. Ans: $[\underbrace{0\,0 \cdots 0}_{N/2}\; N\; \underbrace{0 \cdots 0}_{N/2-1}]$

[3] (c) Recall that the DFT is defined by $\tilde{f}_m = \sum_{j=0}^{N-1} \omega^{-mj} f_j$ where ω is the principal Nth root of 1. State and *prove* the inversion formula that recovers f_j in terms of $\tilde{f}_m$:

for $j = 0, \dots N-1$: $f_j = \frac{1}{N} \sum_{m=0}^{N-1} \omega^{mj} \tilde{f}_m$ is inversion formula: "$1/N$ & sign change."

Prove it: substitute $\tilde{f}_m = \sum_{k=0}^{N-1} \omega^{-mk} f_k$ ← note cannot use index j again!

$$\Rightarrow f_j = \frac{1}{N} \sum_{m=0}^{N-1} \omega^{mj} \sum_{k=0}^{N-1} \omega^{-mk} f_k$$

$$= \sum_{k=0}^{N-1} f_k \cdot \underbrace{\frac{1}{N} \sum_{m=0}^{N-1} \omega^{(j-k)m}}$$ by sum lemma this is 1 when $j = k \mod N$

Ie $j = k$, since $0 \le j, k < N$.

$$= \sum_{k=0}^{N-1} f_k \delta_{jk} = f_j, \quad \forall j$$ $\Box$.

[2] (d) It is easier in practice to deconvolve a signal (or image) that has been blurred by a smooth aperture function or by a discontinuous one? Explain.

Smooth aperture g is much harder since $\hat{g}_m$ decay fast as $|m| \to \infty$, and in deconvolution you must <u>divide</u> Fourier coefficients of signal by $\hat{g}_m$, which become very small as $|m| \to \infty$.

Since signal contains noise, this division amplifies it (a problem), or we must "regularize" and lose all but small $|m|$ information & get poor resolution.

5. [6 points]

[4] (a) Say you want to build an arbitrary-precision reciprocal, that given z to N-digit relative accuracy, can compute $1/z$ to the same relative accuracy. Explain how do it (you may make use of other known algorithms) in the minimum complexity (with respect to N) you can.

$$f(x) = z - \frac{1}{x} \qquad \text{so} \quad f'(x) = \frac{1}{x^2}$$

Newton iteration $\quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{z - 1/x_n}{1/x_n^2}$

$$= (2 - z x_n)\, x_n$$

Initialize $x_0 \in (0, \frac{2}{z})$ although I didn't expect you to remember this.

For the products $z x_n$ & $(\cdots) x_n$ use Strassen's FFT-based convolution scheme, and for subtraction standard arbitrary-precision.

[2] (b) What complexity is your scheme?

Per iteration, Strassen is $O(N \ln N)$ — caveat: assuming const effort per FFT flop.

this dominates over subtraction.

Newton is quadratically convergent so $2^{\#\text{ iterations}} \simeq \#$ digits converged

ie $\#$ iters $= \log_2 N = O(\ln N)$

Complexity $= N \ln N \cdot \ln N = O(N \ln^2 N)$

6. [10 points] Short unrelated questions.

[2] (a) Give the precise definition that a function $f(n)$ has super-algebraic convergence to zero as $n \to \infty$.

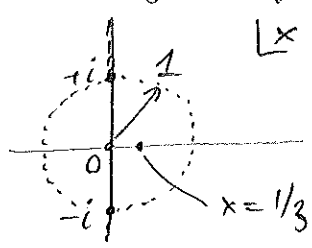
For each $k = 1, 2, \ldots$ $\quad f(n) = O(n^{-k})$

Or, defining the big-O, $\qquad$ { little-o is fine too (here equiv)

$$\forall k = 1, 2, \ldots \ \exists C_k > 0 \ \& \ N_k > 0 \ \text{ st. } \ \forall n > N_k, \ |f(n)| \leq C_k n^{-k}$$

terms is half $2 \cdot 10^6$ since even terms zero.

[3] (b) Up to what power of x do you need to include in the Taylor expansion to $\tan^{-1} x$ to achieve 1000000 digits accuracy at $x = 1/3$? (show working)



By radius of convergence of $\tan^{-1} x$ Taylor series about origin being 1,

rate $r = |x| = 1/3$

Want $(1/3)^n \leq 10^{-1000000}$ $\qquad$ take logs.

$$\Rightarrow \quad n \geq \frac{\ln 10^{-1000000}}{\ln 1/3} = 10^6 \cdot \left(\frac{\ln 10}{\ln 3}\right) \approx 2 \cdot 10^6 \text{ th power of } x.$$

$\frac{\ln 10}{\ln 3} \approx 2$ since $3^2 \approx 10$.

[2] (c) Roughly how many Brent–Salamin iterations do you need to approximate π to 1000000 digits accuracy? (show working)

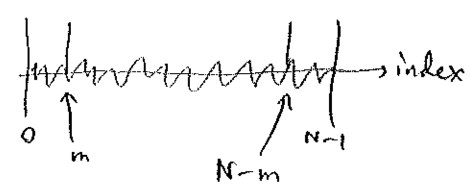
quadratically convergent so $2^n = N = 10^6$ digits

$$n \approx \frac{\ln 10^6}{\ln 2} \approx 6 \left(\frac{\ln 10}{\ln 2}\right) \approx 20.$$

$\frac{\ln 10}{\ln 2} \approx 3$ since $2^3 = 8 \approx 10$.

[3] (d) Filtering. You record a signal vector of length 10^6 of audio sampled at a rate of 10^4 per second. By mistake noise ("hum") at the single frequency of 60 Hz corrupted the recording (this is common). Which mode index/indices should you set to zero in the vector's DFT to remove this noise? $\quad$ } this is a real-world filtering application!

$\tilde{f}$:



60Hz = physical freq $= \frac{m}{T}$

T = total sample time $= \frac{10^6}{10^4} = 10^2$

solve for m: $\quad m = T \cdot 60 = 6000$

Since the single freq. could be any mixture of e^{i6000x} & $e^{-i6000x}$ we should also kill component $N - m = 994000$. (tricky).