Barnett. 4/18/13.

SOLUTIONS

Math 56 Compu & Expt Math, Spring 2013: Midterm 1

4/18/13, pencil and paper, 2 hrs, 50 points. Good luck!

1. [9 points]

if you included $fl(x)$ error on input, messier but same outcome.

[4] (a) Say a computer's algorithm for e^x has relative error in the output of up to ε_{mach}, for $-1 \le x \le 1$. Does this guarantee that the algorithm is *backward stable* in this domain?

We're told $\tilde{f}(x) = e^x(1+\varepsilon_1)$ for $|\varepsilon_1| \le \varepsilon_{mach}$ $\overset{\text{defn.}}{=}$ $f(\tilde{x}) = e^{x(1+\varepsilon)}$ for some $\varepsilon = O(\varepsilon_{mach})$

$= e^x e^{\varepsilon x} \approx e^x(1 + \varepsilon x + \dots)$ $O(\varepsilon^2)$, ignore

Cancel e^x; leaves $\cancel{1} + \varepsilon_1 = \cancel{1} + \varepsilon x$

so $\varepsilon = \dfrac{\varepsilon_1}{x}$ which is not $O(\varepsilon_{mach})$ for $x \to 0$. $\Rightarrow$ No.

[3] (b) Repeat the question for $\sin x$ in the same domain.

Told $(\sin x)(1+\varepsilon_1)$ $\overset{\text{defn bkw stab}}{=}$ $f(\tilde{x}) = \sin(x(1+\varepsilon))$

Taylor or addition thm.

$\approx \sin x + \varepsilon x \cos x + O(\varepsilon^2)$

subtract $\sin x$ from both sides:

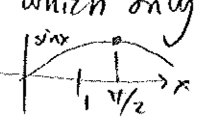
$$\varepsilon_1 \sin x = \varepsilon x \cos x$$

$\Rightarrow$ $\varepsilon = \underbrace{\dfrac{\tan x}{x}}_{\text{bounded in } -1 \le x \le 1} \varepsilon_1$ so $\Rightarrow$ Yes, $\varepsilon = O(\varepsilon_{mach})$ uniformly over all $x \in [-1,1]$.

[2] (c) For some x outside $[-1,1]$ one of the above algorithms cannot be backward stable. Which one, and for what x?

ie with relative error as big as ε_{mach}.

We look for places where small changes in $\tilde{x}$ fail to account for the known relative error in f. Eg. where $f'(x) = 0$, which only happens for $\sin x$ at $x = \frac{\pi}{2}(1+2n)$, eg. $x = \pi/2$

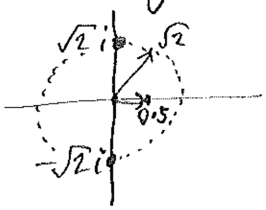


These are where $\mathcal{K}(x) = 0$.

2. [8 points] Consider $f(x) = 1/(2 + x^2)$.

[4] (a) What type, and order/rate, do you expect for convergence of the Taylor series truncated to terms less than x^n, expanding about the origin, when evaluated at $x = 0.5$? Explain

First way: find singularities in $\mathbb{C}$, ie $2+x^2=0$, $x_{sing} = \pm\sqrt{2}\,i$



Thm on Taylor says exponential with rate $\dfrac{\text{center to } x \text{ dist}}{\text{center to } x_{sing} \text{ dist}}$.

so rate $r = \dfrac{0.5}{\sqrt{2}} = \dfrac{1}{2\sqrt{2}}$

OR 2nd way: $f(x) = \frac{1}{2}\left(1+\frac{x^2}{2}\right)^{-1} = \frac{1}{2}\left[1 - \frac{x^2}{2} + \frac{x^4}{2^2} - \cdots \underbrace{\frac{x^n}{2^{n/2}} \cdots}_{\mathcal{E}_n}\right]$ ← n even only.

tail of series $\mathcal{E}_n = \sum_{\substack{k\geq n \\ k\text{ even}}} (\pm)\frac{x^k}{2^{k/2}}$, $|\mathcal{E}_n| \leq \sum_{k\geq n} \frac{x^k}{2^{k/2}} = \left(\frac{x}{\sqrt{2}}\right)^n \sum_{k=0} \frac{x^k}{2^{k/2}} \leq C\left(\frac{x}{\sqrt{2}}\right)^n$

$= C/(2\sqrt{2})^n$

[1] Write an upper bound on the error reflecting this convergence, in big-O notation:

error up to x^n term, $\mathcal{E}_n = O(r^n) = O\left(\frac{1}{(2\sqrt{2})^n}\right)$

[3] (b) Estimate up to what power x^n is needed for this series to reach 16-digit accuracy.

Want $\mathcal{E}_n \approx 10^{-16}$

ie $\dfrac{1}{(2\sqrt{2})^n} \approx 10^{-16}$

ie $n \ln(2\sqrt{2}) \approx \ln 10^{+16}$

$n \approx \dfrac{\ln 10^{16}}{\ln(2\sqrt{2})} = \dfrac{\log_{10} 10^{16}}{\log_{10} 2\sqrt{2}} \approx \dfrac{16}{1/2} \approx 32$

3. [8 points] Consider the "left-sided" finite-difference approximation $f'(x) \approx \frac{f(x)-f(x-h)}{h}$

[5] (a) Derive a rigorous bound on the error that applies to each $h > 0$ [Hint: your bound will need to involve properties of f] about x

suggests Taylor's Theorem, so expand about x w/ remainder.

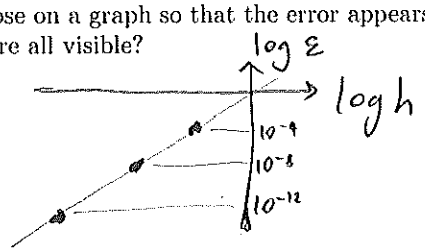
$$\frac{f(x)-f(x-h)}{h} = \frac{\cancel{f(x)} - [\cancel{f(x)} - h f'(x) + \frac{h^2}{2} f''(q)]}{h} \quad \text{some } q \in [x-h, x]$$

$$= \underbrace{f'(x)}_{\text{what we want}} - \underbrace{\frac{h}{2} f''(q)}_{\text{must be the error.}}$$

$$\Rightarrow \text{error } |\varepsilon_h| \le \frac{h}{2} \max_{q\in[x-h,x]} |f''(q)| = O(h)$$

note theorem required $f \in C^2([x-h,x])$

[1] (b) What axes would one choose on a graph so that the error appears as a straight line and yet data at $h = 10^{-4}, 10^{-8}, 10^{-12}$ are all visible?



If you plot ε vs h it's also straight line, but if $h = 10^{-4}$ visible, others are crammed into the origin, invisible.

[2] (c) Explain what happens to the error of the approximation in practice as $h \to 0$

In practice, f is computed to some relative error (maybe $O(\varepsilon_{mach})$) so as h approaches $O(\varepsilon_{mach})$ we get catastrophic cancellation and the approximation becomes terrible.

BONUS Roughly what h has the smallest error? Balancing two sources of error, $O(h)$ from Taylor, & $O\left(\frac{\varepsilon_{mach}}{h}\right)$ from evaluations of f, we get $h \approx \frac{\varepsilon_{mach}}{h}$ so $h \approx \sqrt{\varepsilon_{mach}} \approx 10^{-8}$, say.

4. [7 points] Consider the linear system $\overbrace{\begin{bmatrix} 1 & 0 \\ 10^5 & 1 \end{bmatrix}}^{\text{matrix } A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

How many digits accuracy (relative to the solution norm $\sqrt{x_1^2 + x_2^2}$) are you guaranteed in the solution if the system is solved by a backward stable algorithm with $\varepsilon_{\text{mach}} = 10^{-16}$?

[You may assume a constant of 1 in the backward stability. Hint: full points for rigorous upper bound on the error; generous partial credit for intelligent estimates or other bounds]

ie, lower

tells us to look for upper bound on $\kappa(A)$, but example pairs of vectors $\vec{x}$, $A\vec{x}$ only can give lower bounds. $\Rightarrow$ Need exact κ calc.

We need $\kappa(A) = \|A\| \cdot \|A^{-1}\|$

$$A^TA = \begin{bmatrix} 1 & 10^5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 10^5 & 1 \end{bmatrix} = \begin{bmatrix} 10^{10}+1 & 10^5 \\ 10^5 & 1 \end{bmatrix}$$

$\hookrightarrow$ eigvals

$$\lambda^2 - (10^{10}+2)\lambda \cancel{- 10^{10}} - 1 \cancel{+ (10^5)^2} = 0 \qquad \text{so } \lambda = \frac{1}{2}\left(10^{10}+2 \underset{\substack{\uparrow \\ + \text{ gives } \lambda_{max}}}{\pm} \sqrt{(10^{10}+2)^2 - 4}\right)$$

yuk, can we bound it?

use $\sqrt{(10^{10}+2)^2 - 4} \le 10^{10}+2$

so $\lambda_{max} \le 10^{10}+2$ $\qquad$ so $\|A\| = \sqrt{\lambda_{max}(A^TA)} \le \sqrt{10^{10}+2}$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 & \\ -10^5 & 1 \end{bmatrix}$$

which gives identical calc as before $\Rightarrow \|A^{-1}\| \le \sqrt{10^{10}+2}$

So $\kappa(A) \le 10^{10}+2$

Backward Stability Theorem: $\quad$ rel. err $\le \underset{\approx 10^{10}}{\kappa} \cdot \underbrace{O(\varepsilon_{mach})}_{10^{-16} \text{ if const}=1}$

so rel. err $\lessapprox 10^{-6}$

6-digit accuracy

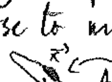
Hard! Makes you think. $\leftarrow$ [BONUS] Find a right-hand side **b** for which the above worst-case prediction is (nearly) achieved.

$\kappa(A)$ is a worst-case value of $\kappa(\vec{b})$ for the linear solve problem $\vec{x} = \vec{f}(\vec{b}) = A^{-1}\vec{b}$, over all $\vec{b}$. We need to rederive κ :

$$\kappa(\vec{b}) = \max_{\delta\vec{b} \neq \vec{0}} \frac{\|\delta f\| / \|f\|}{\|\delta b\| / \|b\|} = \underbrace{\max_{\delta\vec{b} \neq \vec{0}} \frac{\|A^{-1}\delta b\|}{\|\delta b\|}}_{=: \|A^{-1}\|, \text{ fixed}} \cdot \underbrace{\frac{\|b\|}{\|A^{-1}b\|}}$$

$\Rightarrow$ RHS-dependent part: want to maximize $\Rightarrow$ A^{-1} must shrink $\vec{b}$ the most, ie A must grow $\vec{x}$ the most.

Easy to find $\vec{x}$ which close to max growth:



$$\begin{bmatrix} 1 & 0 \\ 10^5 & 1 \end{bmatrix} \underset{\vec{x}}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} 1 \\ 10^5 \end{bmatrix}}$$

so, this RHS does it.

Note, strangely, a 'typical' $\vec{b}$ will give $\vec{x} = A^{-1}\vec{b}$ about $10^5\times$ larger, (not smaller), so rel. err $\simeq O(\varepsilon_{mach})$!

5. [7 points] Given $y > 0$, you wish to approximate $x = \sqrt{y}$ using elementary operations.

[1] (a) Derive a Newton iteration that converges to the desired x [Hint: x must be a root of something]

$f(x)$

As in class pick $f(x) = x^2 - y$

not $x - \sqrt{y}^2$ which demands computing $\sqrt{y}$!

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - y}{2x_n}$$

$$= \frac{x_n}{2} + \frac{y}{2x_n}$$

[3] (b) Derive a big-O estimate on the error ε_n after n iterations.

I also gave 2/3 for instead proving this.

This is hard, to make you think.

Recall Newton has 'quadratic' convergence, ie $\varepsilon_{n+1} \le C \varepsilon_n^2$

Start at ε_0:

$$\varepsilon_1 \le C \varepsilon_0^2$$

$$\varepsilon_2 \le C \varepsilon_1^2 \le C(C\varepsilon_0^2)^2 = C^3 \varepsilon_0^4$$

$$\varepsilon_3 \le C(C(C\varepsilon_0^2)^2)^2 = C^7 \varepsilon_0^8$$

etc.

See by induction that $\varepsilon_n \le C^{2^n-1} \varepsilon_0^{2^n} = \frac{1}{C}(C\varepsilon_0)^{2^n}$

so $\varepsilon_0 = O\left(c^{2^n}\right)$ for some const $c < 1$

we hope! (relies on good starting guess, ε_0 small).

6. [11 points] Short answers.

we assume $N \to \infty$ (forgot to say this).

[3] (a) Prove whether $\underbrace{N + \frac{N}{(\log_{10} N) - 7}}_{f(N)} = O(\underbrace{N}_{g(N) = N})$, giving, if true, a constant and corresponding condition on N.

$$\frac{f(N)}{g(N)} = 1 + \underbrace{\frac{1}{\log_{10} N - 7}}_{\text{only bounded once } \log_{10} N > 7} \overset{?}{\leq} C$$

holds $\forall N \geq 10^8$ with $C = 2$

[2] (b) Prove whether $\underbrace{\sqrt{1+x^2}\sin x}_{f(x)} = o(\underbrace{x}_{g(x)})$ as $x \to \infty$

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \underbrace{\boxed{\frac{\sqrt{1+x^2}}{x}}}_{\text{has limit} = 1} \cdot \underbrace{\sin x}_{\text{has no limit (despite being bounded)}} \Rightarrow \text{no.}$$

[2] (c) How close to 1 does x have to be such that the relative condition number of computing $\underbrace{\sqrt{x-1}}_{f(x)}$ is 10^8?

$$\kappa(x) = \left|\frac{x f'(x)}{f(x)}\right| = \left|\frac{x(x-1)^{-1/2}/2}{\sqrt{x-1}}\right| = \left|\frac{x}{2(x-1)}\right|$$

Set $\kappa \approx 10^8$:

$\underbrace{|x|}_{\approx 1} = 2 \times 10^8 |x-1|$ so $|x-1| \approx \frac{1}{2 \times 10^8}$ ie 5×10^{-9} distance away.

[2] (d) Prove that $\|A^{-1}\| = \left(\min_{x \neq 0} \frac{\|Ax\|}{\|x\|}\right)^{-1}$

now pick $\vec{x}' = A^{-1}\vec{y}$.

$$\|A^{-1}\| := \max_{\vec{y} \neq \vec{0}} \frac{\|A^{-1}\vec{y}\|}{\|\vec{y}\|} = \max_{\vec{x}' \neq \vec{0}} \frac{\|\vec{x}'\|}{\|A\vec{x}'\|} = \left(\min_{\vec{x}' \neq \vec{0}} \frac{\|A\vec{x}'\|}{\|\vec{x}'\|}\right)^{-1}$$

since min of the reciprocal is reciprocal of the max.

[2] (e) Give a definition of an algorithm $\tilde{f}(x)$ for a problem $f(x)$ being *stable*:

For each x in some domain of interest, there exists $\tilde{x} = x(1+\varepsilon)$ for some $\varepsilon = O(\varepsilon_{mach})$, such that $\frac{\|\tilde{f}(x) - f(\tilde{x})\|}{\|f(\tilde{x})\|} = O(\varepsilon_{mach})$.

"nearly right answer to nearly right question"!

Or, for each x, $\exists\, \varepsilon = O(\varepsilon_{mach})$ s.t. $\|\tilde{f}(x) - f(x(1+\varepsilon))\| \leq \|f(x(1+\varepsilon))\| O(\varepsilon_{mach})$