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MATH 56 WORKSHEET : DFT basics.

$$\boxed{F_{mj} = \omega^{-mj}, \qquad \omega = e^{2\pi i/N}}$$

A) Write out F for $N=4$:

$$F = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

Use this to compute the DFT of :

i) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ ← const signal.

ii) $\begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix}$ ← frequency-1 complex exponential

B) Prove that if A is a unitary matrix, it preserves length [Hint : $x^* I x$]

C) Prove the inverse DFT formula : $f_j = \frac{1}{N} \sum_{m=0}^{N-1} \omega^{jm} \tilde{f}_m$
by substituting in $\tilde{f}_m = \sum_{k=0}^{N-1} \omega^{-mk} f_k$, swapping sum order, simplifying :

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$$\boxed{F_{mj} = \omega^{-mj}, \qquad \omega = e^{2\pi i/N}}$$

SOLUTIONS

A) Write out F for $N=4$:

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

since $\omega = e^{i\pi/2} = i$.

Use this to compute the DFT of:

i) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \xrightarrow{\text{DFT}} \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

const signal.

ie $\tilde{f}_0 = 4$ others zero.

ii) $\begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} \xrightarrow{\text{DFT}} \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \end{bmatrix}$ ← $\tilde{f}_1 = 4$ is the only nonzero

frequency-1 complex exponential

B) Prove that if A is a unitary matrix, it preserves length [Hint: $x^* I x$]

$A^* A = I$

$$\|x\|^2 = x^* x = x^* I x = x^* A^* A x = (Ax)^* Ax = \|Ax\|^2$$

$\forall x \in \mathbb{C}^N$

C) Prove the inverse DFT formula: $f_j = \frac{1}{N}\sum_{m=0}^{N-1} \omega^{jm} \tilde{f}_m$ ← supposed inverse of DFT.

by substituting in $\tilde{f}_m = \sum_{k=0}^{N-1} \omega^{-mk} f_k$, swapping sum order, simplifying:

$$f_j = \frac{1}{N}\sum_{m=0}^{N-1} \omega^{jm} \sum_{k=0}^{N-1} \omega^{-mk} f_k = \sum_{k=0}^{N-1} f_k \cdot \underbrace{\frac{1}{N}\sum_{m=0}^{N-1} \omega^{m(j-k)}}$$

sum lemma $\to \begin{cases} 1 & j-k = 0 \pmod N \\ 0 & \text{otherwise} \end{cases}$

The only time $j = k \pmod N$ is when $j=k$.

$\Rightarrow$ expression $= \sum_{k=0}^{N-1} f_k \delta_{kj} = f_j$

Holds $\forall j = 0, \dots N-1$

$\Rightarrow$ We proved it's the inverse. $\square$