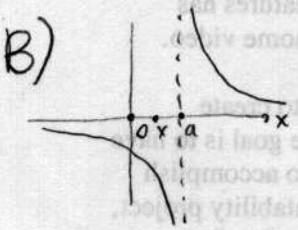
3/26/1[illegible]
Barnett

MATH 56 WORKSHEET : Asymptotics & Convergence Types

A) Is $\frac{\log n}{n} = O(n^{-1/2})$ as $n \to \infty$? [prove it]

Say $C > 0$ is fixed. Is $C^{-n} = o\left(\frac{1}{n!}\right)$ as $n \to \infty$?

B) 

Let $a > 0$. Write the Taylor series for $f(x) = \frac{1}{x-a}$ about $x=0$:

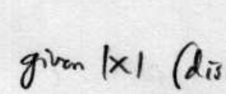
[Hint: $\sum_{k=0}^{\infty} \left(\frac{x}{a}\right)^k = ?$]

Fixing x, derive a bound on ε, the error in n-term Taylor series; writing in O notation:

Make a conjecture on convergence <u>rate</u> given $|x|$ (dist. from expansion pt), a (dist. of singularity from expansion pt)

C) What axes show algebraic convergence as <u>straight</u> line?

How about exponential conv.?



← Interpret the slopes.

BONUS: What if ε shrinks so fast as to double the # correct <u>digits</u> as $n \to n+1$?

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$\longrightarrow$ SOLUTIONS $\longleftarrow$

A) Is $\underbrace{\frac{\log n}{n}}_{f(n)} = O(\overbrace{n^{-1/2}}^{g(n)})$ as $n \to \infty$? [prove it]

$$\frac{f(n)}{g(n)} = \frac{\log n \to \infty}{n^{1/2} \to \infty} \xrightarrow{\text{l'Hôpital}} \frac{n^{-1}}{\frac{1}{2}n^{-1/2}} = \frac{1}{2\sqrt{n}} \to 0 \quad \text{so yes}$$

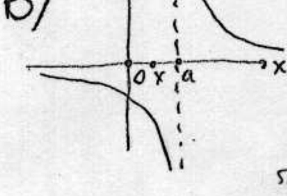
(it's even little-o).

Say $C>0$ is fixed. Is $C^{-n} = o\left(\frac{1}{n!}\right)$ as $n \to \infty$?

$$\frac{f}{g} = \frac{C^{-n}}{1/n!} = \frac{n!}{C^n} \neq 0 \Rightarrow \text{no}$$

can't use l'Hôp since $\frac{d}{dn}(n!) = ??$ Use: $n!$ grows faster than exponential.

B)



Let $a>0$. Write the Taylor series for $f(x) = \frac{1}{x-a}$ about $x=0$:

[Hint: $\sum_{k=0}^{\infty} \left(\frac{x}{a}\right)^k = ?$] $\rightarrow$ $\frac{1}{1-x/a} = \frac{a}{a-x}$ $\leftarrow$ is $(-a)$ times what we want

so $f(x) = -\frac{1}{a}\sum_{k=0}^{\infty}\left(\frac{x}{a}\right)^k = -\frac{1}{a}\left[1 + \frac{x}{a} + \frac{x^2}{a^2} + \cdots\right]$

$= -\frac{1}{a} - \frac{x}{a^2} - \frac{x^2}{a^3} - \cdots$

Fixing x, derive a bound on ε, the error in n-term Taylor series; writing in O notation:

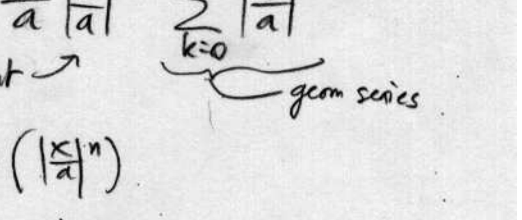
$$\varepsilon = -\sum_{k>n} \frac{x^k}{a^{k+1}} \quad \text{so} \quad |\varepsilon| \le \frac{1}{a}\sum_{k>n}\left|\frac{x}{a}\right|^k = \frac{1}{a}\underbrace{\left|\frac{x}{a}\right|^{n+1}}_{\text{brought out}}\underbrace{\sum_{k=0}\left|\frac{x}{a}\right|^k}_{\text{geom series}}$$

$$= \underbrace{\frac{1}{a}\,\frac{1}{1-|x/a|}}_{\text{indep. of } n}\left|\frac{x}{a}\right|^{n+1} = O\left(\left|\frac{x}{a}\right|^{n+1}\right) = O\left(\left|\frac{x}{a}\right|^{n}\right)$$

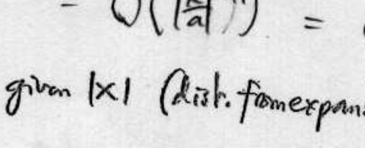
Make a conjecture on convergence rate given $|x|$ (dist. from expansion pt), a (dist. of singularity from expansion pt):

$$\text{rate } r = \frac{|x|}{a} = \frac{\text{dist from expansion pt}}{\text{singularity dist from expansion pt}}$$

C) What axes show algebraic convergence as straight line?



How about exponential conv.?



$\leftarrow$ Interpret the slopes.

BONUS: What if ε shrinks so fast as to double the # correct digits as $n \to n+1$?