The Fractal Nature of Schottky Groups

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Abstract

In this paper we study the limit sets of Schottky groups composed of two Mobius maps a and b and their inverses A and B. When composed together to create infinite "words" such as aBBaBAbAbabaB$\cdots$ these maps form limit sets whose nature can be determined based on certain properties of the groups. The fractal images seen below represent approximations of the limit sets of Schottky groups with carefully chosen parameters.

1 Mobius Maps

A Mobius map T takes

$$z \to \frac{az+b}{cz+d},$$

where $z, a, b, c, d \in \mathbb{C}$. Let a be one such map and let A be its inverse, that is, $aA(z) = Aa(z) = z$. We can think of a as the 2×2 matrix

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and indeed the composition of two Mobius maps is given by the product of their matrix representations. The maps we are interested in are the maps that pair two circles together: given two circles A and a, the map takes the outside of A to the inside of a (see figure 1).

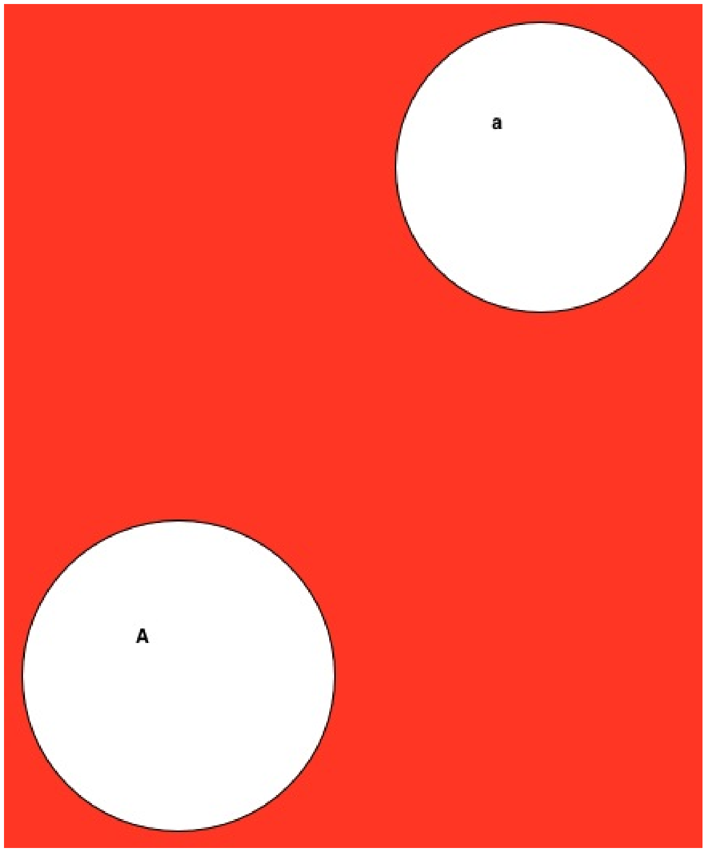

Figure 1: We can construct a Mobius map that takes the outside of A (the region in red plus a) to the inside of a. The red region extends to ∞ (we are on the complex plane).

In figure 2 we see the effect of repeated application of the map a.

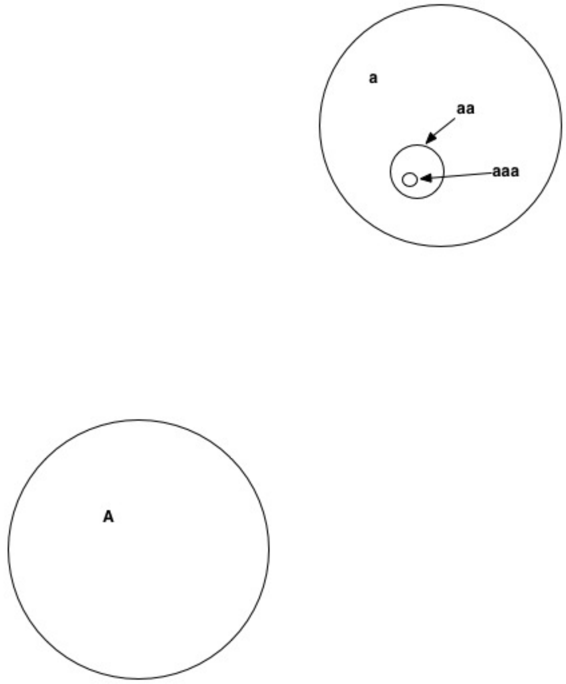

Figure 2: The map a takes the Schottky circle a deeper inside itself.

Taking two Mobius maps a and b, whose circle pairs do not overlap, we calculate their inverses as A and B. Repeatedly applying these maps to a point maps the whole complex plane into one of these four circles. In figure 3 we show the image circles of depth 1 and depth 2.

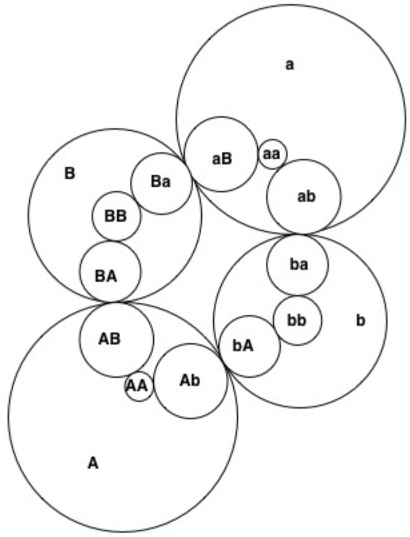

Figure 3: Two pairs of tangent Schottky circles.

The points that result from all infinite combinations of a, b, A, and B are called the limit set of the generators a and b, and the set of all the combinations themselves is called a Schottky group. In figure 3, the limit set of the group will be a continuous line running through the tangent points of all the circles.

The fractal images in this paper are all the limit sets of specially chosen Schottky groups. We must go over some complex arithmetic to understand how they are formed. In complex arithmetic, division by 0 is allowed, with

$$\pm\frac{z}{0} = \infty$$

for $z \in \mathbb{C}, z \neq 0$. ∞ is further defined by $z \pm \infty = \infty$, i.e. $-\infty$ does not exist. Thus a Mobius map can take any point in the complex plane to any other point, with ∞ as one "point". By setting a, b, c, d as desired, a Mobius map can represent any translation $T(z) = z + a$, any rotation $T(z) = kz$ with $|k| = 1$, and any scaling $T(z) = kz$, $|k| > 1$. It turns out that any Mobius map T is conjugate to one of these three transformations, where $\hat{T} = RTR^{-1}$ is the conjugate of T under some transformation R.

Every Mobius map can be classified as one of three types: loxodromic, parabolic, or elliptic. A parabolic map is conjugate to a translation, an elliptic map is conjugate to a rotation, and a loxodromic map is conjugate to a scaling. Parabolic maps, being translations, have the special property that they have only one fixed point. A fixed point of a Mobius map is any point $z \in \mathbb{C}$ such that $T(z) = z$. If $T = kz$, then the fixed points of T are 0 and

∞, since $T(0) = 0$ and $T(\infty) = \infty$. The fixed points of a transformation are given by the solutions to the equation

$$z = \frac{az + b}{cz + d}.$$

A parabolic transformation has only the fixed point $z = \infty$. The above equation has only one solution only when $a + d = \mathrm{Tr}\ T = \pm 2$, where $\mathrm{Tr}\ T$ is the trace of T. If we let a and b be parabolic and let their circle pairs be tangent, the composite map $abAB$ has only one fixed point at the tangent between the circles a and b. If we specify in addition that $\mathrm{Tr}\ abAB = -2$, then the limit set of the Schottky group generated by a and b is a continuous curve.

The limit sets of Shottky groups can be found by using a depth-first search (DFS) algorithm. Of course we cannot explore all the infinite compositions of a, b, A, and B, but if we set a high enough depth then the result is the limit set as far as we can tell. The images below were generated using the DFS algorithm outlined on page 148 of *Indra's Pearls*.

2 The Apollonian Gasket

Somewhat surprisingly, the Apollonian gasket can be generated by finding the limit set of a parabolic Schottky group. The circle pairs of this group can be seen in figure 4.

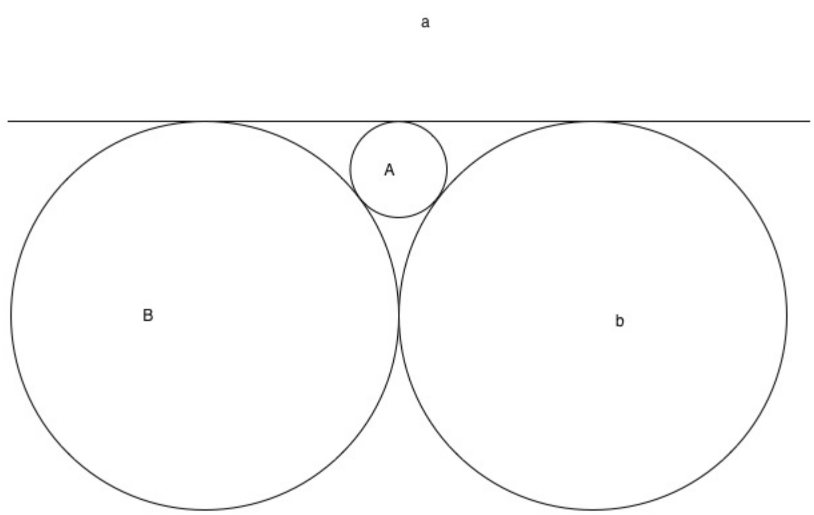

Figure 4: The Schottky circles of the group used to generate the Apollonian gasket. In the complex plane a straight line is just a circle through ∞.

We generated figures 5 and 6 by using the DFS algorithm on the generators

$$a = \begin{pmatrix} \sqrt{2} & i \\ -i & \sqrt{2} \end{pmatrix}, \quad b = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}.$$

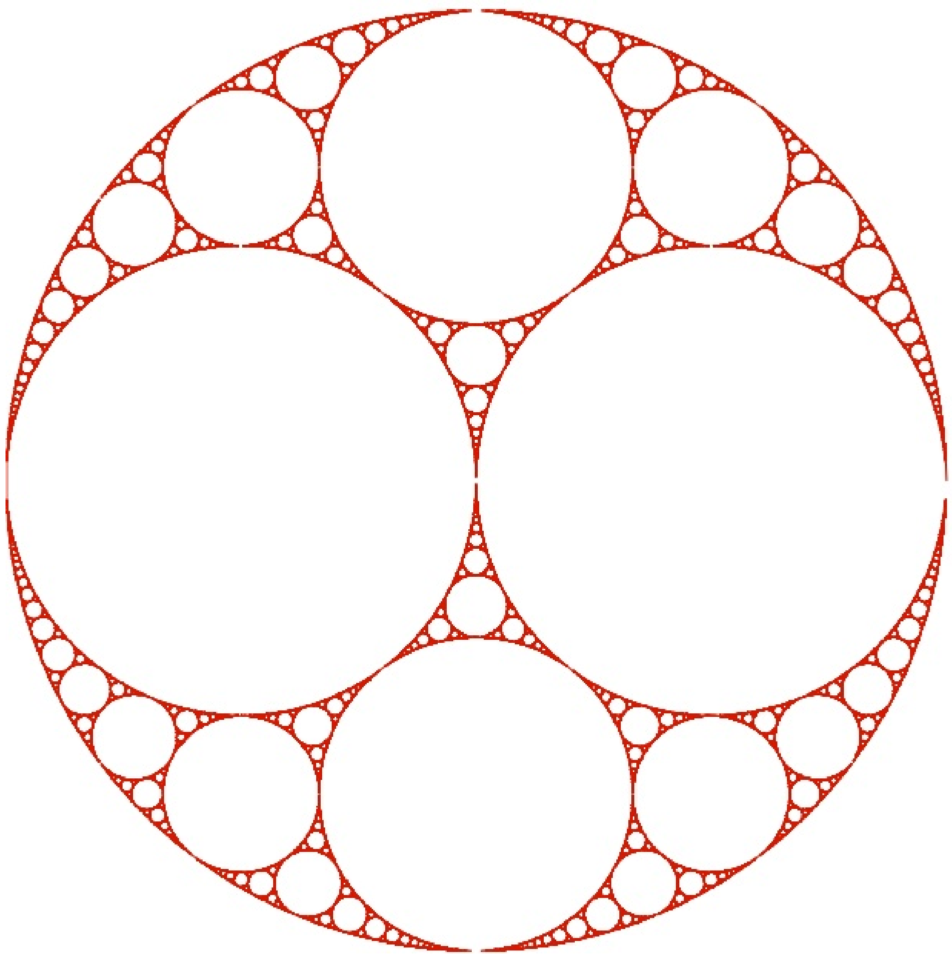

Figure 5: The Apollonian gasket generated by a Schottky group.

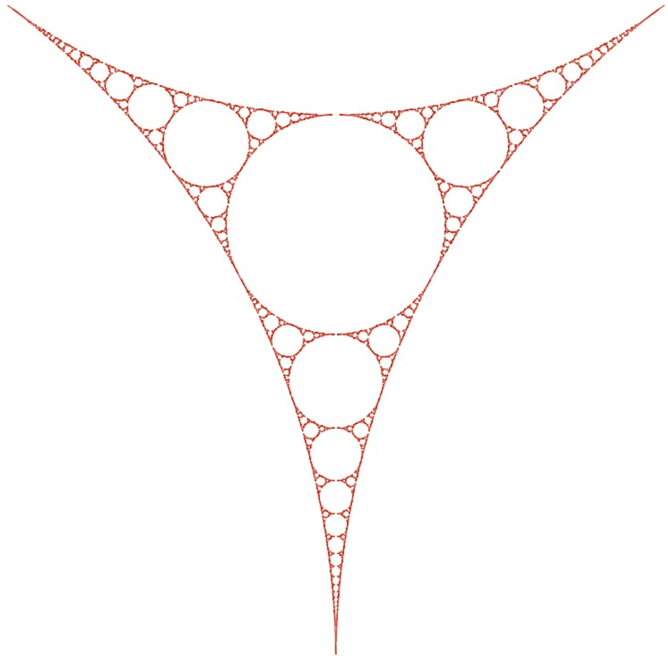

Figure 6: A close-up view of the gasket.

3 The Significance of Trace

The generators a and b can be completely determined by Tr a, Tr b, and Tr ab. We can solve for Tr ab from the first two, so by setting Tr a and Tr b we can create completely new generators. Figures 7 and 8 were generated by Tr $a = 1.87 + 0.5i$, Tr $b = 1.87 - 0.5i$ and Tr $a = 1.87 + 0.1i$, Tr $b = 1.87 - 0.1i$. Since they have complex traces these groups are loxodromic. Smaller imaginary parts lead to tighter spirals, as shown by the difference between figure 7 and figure 8.

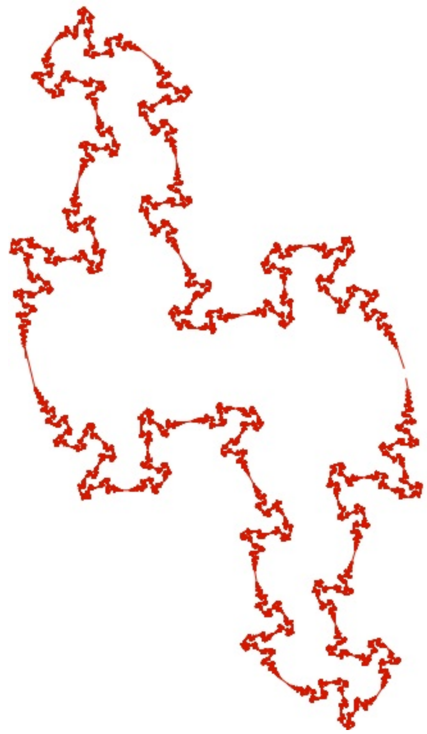

Figure 7: Jordan curve generated by Tr $a = 1.87 + 0.5i$, Tr $b = 1.87 - 0.5i$.

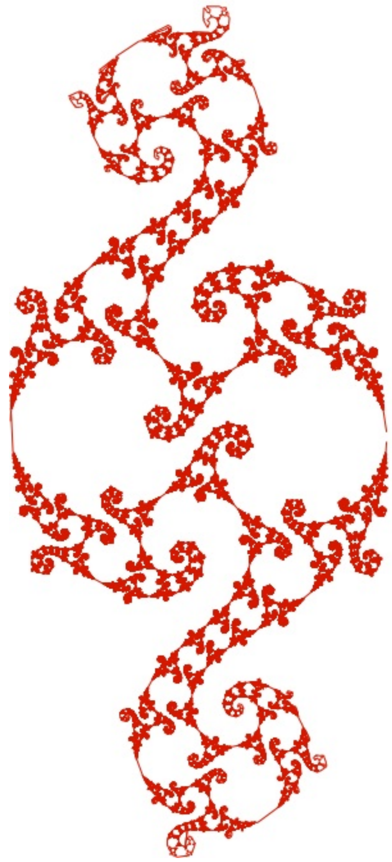

Figure 8: Jordan curve generated by Tr $a = 1.87 + 0.1i$, Tr $b = 1.87 - 0.1i$.

4 References

Mumford, David, Caroline Series, and David Wright. *Indra's Pearls: The Vision of Felix Klein*. Cambridge, U.K.: Cambridge University Press, 2002. Print.