TOPOLOGY: HOMEWORK 7

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1. Let X and Y be topological spaces and $\mathscr{B}_Y$ a basis for the topology on Y. Without using Thm 10.9 (i.e., using only the definition), prove that a function $f : X \to Y$ is continuous if and only if $f^{-1}(B)$ is open in X for every $B \in \mathscr{B}_Y$.

2. Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = \begin{cases} 1 - x^2 - y^2 & x^2 + y^2 \leq 1 \\ x^2 + y^2 - 1 & x^2 + y^2 \geq 1 \end{cases}$ is continuous.

   (As a starting point, prove that polynomials in two variables are continuous.)

3. (Weak Topologies) The motivation behind a topology is that it is the minimal structure to make sense of continuous functions. In this problem, we will motivate the definition of the product topology in this context.

   Let X_1 and X_2 be topological spaces and consider the projection maps $\pi_1 : X_1 \times X_2 \to X_1$ and $\pi_2 : X_1 \times X_2 \to X_2$. Prove that the product topology $\mathscr{T}_{X_1 \times X_2}$ on $X_1 \times X_2$ is the *coarsest* topology on which π_1 and π_2 are continuous.[1]

4. (Stereographic Projection) Define a function $f : S^1 \setminus \{(0, 1)\} \to \mathbb{R}$ as follows:

   - Let $p \in S^1 \setminus \{(0, 1)\}$.
   - Consider the ray r_p starting at $(0, 1)$ and passing through the point p.
   - Let $(x_p, 0)$ be the point where r_p intersects the x-axis. Define $f(p) = x_p$.

   This map is the stereographic projection of the circle onto the real line.

   (a) Prove that f is well-defined (i.e., f is a function).

   (b) Give an explicit definition of f.

   (c) Prove that f is a homeomorphism. To show that f is open, it is enough to argue descriptively. That is, no formal write-up is required but an explanation should be provided.

   Hint: To show the continuity of f, consider the explicit definition from part (b). Extending the domain to $\{(x, y) \in \mathbb{R}^2 \mid y < 1\}$, the function f is still defined. Prove that this function is continuous and make the desired conclusion.

5. Prove that the composition of two homeomorphisms is a homeomorphism.

6. (Locally Constant) A function $f : X \to Y$ is *locally constant* if, for every $x \in X$, there is a neighborhood U of x such that $f|_U$ is a constant function.

   Let X be a topological space. Prove that f is locally constant if and only if f is continuous when Y is given the discrete topology.

[1]This construction generalizes substantially: given functions $f_\alpha : Y \to X_\alpha$ from a set Y to topological spaces X_α for $\alpha \in I$, there is a coarsest topology, known as the **weak topology**, on Y which makes the functions f_α continuous.