1. (Classification of 1-manifolds, part 2) Continuing from HW11#2, these problems will almost complete the classification of 1-manifolds. Let M be a 1-manifold with atlas

$$\mathscr{A} = \{(\varphi, U) \mid \varphi : U \to (0,1) \text{ is a homeomorphism}\}$$

and $(\varphi, U), (\psi, V) \in \mathscr{A}$.

(a) Assume that M is connected and $U \cap V$ has two connected components. Prove that M is homeomorphic to S^1.

Hint:

i. Let W_0 and W_1 be the connected components of $U \cap V$. Show that (φ, U) and (ψ, V) overlap. Apply HW11#2(c) and argue that we may assume $\varphi(W_0)$ and $\psi(W_0)$ are lower and that $\varphi(W_1)$ and $\psi(W_1)$ are upper.

ii. Write

$$\varphi(W_0) = (0, a),\ \varphi(W_1) = (a', 1), \qquad \psi(W_0) = (0, b),\ \psi(W_1) = (b', 1).$$

Let S be the boundary of $[0,1] \times [0,1]$. That is, $S = (\{0,1\} \times [0,1]) \cup ([0,1] \times \{0,1\})$. Define a function $f : [0,1] \to S$ by a piecewise linear map so that

$$f(0) = (0,0),\ f(a) = (1,0),\ f(a') = (1,1),\ f(1) = (0,1).$$

Define $g : [b, b'] \to S$ linearly by

$$g(b) = (0,0),\ g(b') = (0,1).$$

Finally, define $\eta : U \cup V \to S$ by $\eta(x) = \begin{cases} f \circ \varphi(x) & x \in U \\ g \circ \psi(x) & x \in V \setminus U \end{cases}$. Prove that η is a homeomorphism of $U \cup V$ and S.

iii. Using (ii), show that $U \cup V$ is compact. Using the connectedness of M, conclude that η is a homeomorphism of M and S.

(b) Assume (φ, U) and (ψ, V) overlap and that $U \cap V$ is connected. Prove that $U \cup V$ is homeomorphic to $(0,1)$.

Hint: Let $W = U \cap V$. Applying HW11#2(c), assume that $\varphi(W)$ and $\psi(W)$ are upper. Let $\psi(W) = (b, 1)$. Define $\eta : U \cup V \to (0,1)$ by $\eta(x) = \begin{cases} \varphi(x) & x \in U \\ 1 + b - \psi(x) & x \in V \setminus U \end{cases}$.

2. Determine which compact surface has word $abd^{-1}cab^{-1}d^{-1}c$.

3. Consider the Latin alphabet

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

Partition these characters (considered as topological spaces) into sets in two ways:

(a) by homeomorphism (i.e., the spaces are pairwise homeomorphic) and

(b) by homotopy equivalence.

Note: No explicit maps or justification are required.

4. Let $p_1, p_2, p_3 \in S^2$ be distinct points and consider the thrice-punctured sphere

$$X = S^2 \setminus \{p_1, p_2, p_3\}\,.$$

Deform X until it is easy to describe and call the result Y. Choose a base point y_0 and describe the "essential" loops for the fundamental group $\pi_1(Y, y_0)$.