1. Let M_1 be an n-manifold and M_2 an m-manifold. Prove that $M_1 \times M_2$ is a $(n+m)$-manifold.

2. (Classification of 1-manifolds, part 1) Let

$$\mathscr{A} = \{(\varphi, U) \mid \varphi : U \to (0,1) \text{ is a homeomorphism}\}$$

be an atlas on a 1-manifold M and let $(\varphi, U), (\psi, V) \in \mathscr{A}$.

IMPORTANT: This is an atlas because $\mathbb{R}$ is homeomorphic to $(0,1)$. However, taking $(0,1)$ to be the codomain of our charts will make this problem easier to work through.

(a) Assume $U \cap V \neq \varnothing$ and $U \setminus V \neq \varnothing$. Prove that if $\{x_n\}_{n=1}^{\infty}$ is a sequence in $U \cap V$ converging to $x \in U \setminus V$ then $\{\psi(x_n)\}_{n=1}^{\infty}$ has no limit in $\psi(V)$.

Hint: Use the fact that M is Hausdorff. An old result from class will help.

(b) Let $I \subset (0,1)$ be a proper open subinterval. Then I is $\begin{cases} \textit{upper} & \text{if } I = (a,1) \\ \textit{lower} & \text{if } I = (0,b) \end{cases}$ where $0 < a$ and $b < 1$. In either case, I is called *outer*. Prove that I is outer if and only if there is a sequence in I which doesn't converge in $(0,1)$.

(c) We say that (φ, U) and (ψ, V) *overlap* if $U \cap V \neq \varnothing$, $U \setminus V \neq \varnothing$, and $V \setminus U \neq \varnothing$.

Assume that (φ, U) and (ψ, V) overlap and let W be a connected component of $U \cap V$. Prove that $\varphi(W)$ and $\psi(W)$ are outer.

Hint: First show that $\varphi(W)$ is a proper subinterval of $\varphi(U) = (0,1)$. Using (a) or the fact that manifolds are locally connected, show that $\varphi(W)$ is an open interval. By symmetry of the argument, $\psi(W)$ will also be a proper open interval. Using the characterization in (b), construct a sequence in $\varphi(W)$ and use (a) again to show that $\psi(W)$ is outer. Conclude that $\varphi(W)$ must also be outer.

(d) Using (c), conclude that $U \cap V$ has at most two connected components for any two charts (φ, U) and (ψ, V).

3. Assume $q : X \to Y$ is a surjective continuous function. Prove that if q is an open function then q is a quotient map.[1]

4. Let X be a T_4 space. Prove that if A is closed in X then $(A, \mathscr{T}_A)$ is T_4.

5. Let X be a T_3 space and $A \subset X$ closed. Prove that the quotient space X/A is Hausdorff.

[1] This result holds if q is, instead, a closed function.