TOPOLOGY: THE CANTOR SET

DATE: AUGUST 2, 2016

This assignment is about the Cantor Set, a remarkable subset of $[0,1]$. Named for the mathematician Georg Cantor, this set is a fractal (a type of self-similar object) and possesses many strange properties. Solutions for the problems on the following page are due **August 24, 2016**. Unlike standard assignments, **groups of up to 3 people may submit a single assignment for credit. For each problem, list who worked on that problem**. (This will *not* affect scores.)

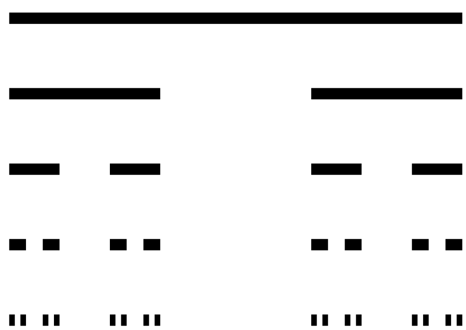
Description 1: To construct the Cantor set, we need to apply a recursive process to the interval $[0,1]$. Let $F_0 = [0,1]$. We obtain F_1 by removing the middle third of closed line segments:

$$F_1 = [0,1] \setminus (\tfrac{1}{3}, \tfrac{2}{3}) = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1].$$

Now we repeat this process to obtain F_2:

$$F_2 = F_1 \setminus \big((\tfrac{1}{9}, \tfrac{2}{9}) \cup (\tfrac{7}{9}, \tfrac{8}{9})\big) = [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1].$$

Repeating this, we get a collection $\{F_n\}_{n=0}^{\infty}$ of sets. Visually, F_0 through F_4 appear as follows:



Finally, the Cantor set is defined to be the intersection of these sets: $\boxed{C = \bigcap_{n=0}^{\infty} F_n}$.

We know that $C \neq \varnothing$ because the endpoints of the removed intervals remain. That is, points such as $\frac{1}{3} \in C$ since, after the interval $(\frac{1}{3}, \frac{2}{3})$ is removed, $\frac{1}{3}$ is in the top third of an interval forever after.

However, not every point left over is the endpoint of some interval. For instance, $\frac{1}{4} \in C$ since $\frac{1}{4}$ alternates between being in the bottom third and the top third of intervals.

Description 2: Alternatively, we may think of the Cantor set as the points in $[0,1]$ whose *ternary* expansion has no ones. Every number in $[0,1]$ can be written as $0.x_1x_2x_3\ldots$ where $x_i \in \{0,1,2\}$. This corresponds to "choosing" the left (0), middle (1), or right (2) third of the interval specified by the previous choice. So $\frac{1}{4} = 0.020202\ldots \in C$.

For instance, $x = 0.2x_2x_3\ldots$ means that $x \in [\frac{2}{3}, 1]$. Further specifying that $x_2 = 0$ forces $x = 0.20x_3\ldots$ to be in the interval $[\frac{2}{3}, \frac{7}{9}]$.

In what follows, either description of C may be used. Some properties are most easily proved using one definition instead of the other.

Prove at least four of the following (extra credit for each additional solution):

1. C is closed. Conclude that C is compact.
2. $\operatorname{Int} C = \varnothing$. Conclude that C is nowhere dense (i.e., $\operatorname{Int} \overline{C} = \varnothing$).
3. Every point of C is a limit point of C. Conclude that no point of C is an isolated point.
4. The set E, consisting of endpoints of the intervals removed to obtain C, is countable. For instance, $\frac{1}{3} \in E$ since $\left(\frac{1}{3}, \frac{2}{3}\right)$ was removed in the first step.
5. C is uncountable.
6. The sum of the lengths of intervals removed from $[0, 1]$ is equal to 1. (For an interval (a, b), the length $\ell((a, b)) = b - a$.)
7. C is totally disconnected (i.e., the only connected components are singleton sets).

These are not a complete list of the interesting (and seemingly contradictory) properties of the Cantor set:

- Using C, one can define the Cantor function (also known as the Devil's Staircase), a non-decreasing surjective continuous function $f : [0, 1] \to [0, 1]$ whose derivative is 0 (wherever $f'(x)$ exists).
- C is a complete metric space.
- C is an example of an uncountable set with Lebesgue measure 0.
- For real numbers, we can "sum" sets: $A + B = \{a + b \mid a \in A, b \in B\}$. The surprising fact is that $C + C = [0, 2]$. (Yes, the *entire* interval.)
- Above we proved that C is: totally disconnected, perfect (closed with no isolated points), compact, and (being a subset of $[0, 1]$) a metric space. Any nonempty set with these properties is necessarily homeomorphic to C.